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THE FUNDAMENTAL PROBLEM OF PRACTICAL STATISTICS.

BY KARL PEARSON, F.R.S.

(1) SOME excuse must be given for once more returning to a problem which is as ancient as Bayes, has been apparently treated from most aspects by Laplace, and has been criticised and re-criticised by Boole, De Morgan, Venn and Edgeworth. The problem I refer to is that of "inverse probabilities" and in practical statistics it takes the following form:

An "event" has occurred p times out of $p+q=n$ trials, where we have no *a priori* knowledge of the frequency of the event in the total population of occurrences. What is the probability of its occurring r times in a further $r+s=m$ trials?

In statistical language a first sample of n shows p marked individuals, and we require a measure of the accordance which future samples are likely to give with this result. For example, a medical treatment is found to be successful in p out of n cases, we require some measure of the probable stability of this ratio. It is on this stability and its limits according to the size of the first sample that the whole practical doctrine of statistics, which is the theory of sampling, actually depends. We usually state the "probable errors" of results without visualising the strength or weakness of the logic behind them, and without generally realising that if the views of some authors be correct our superstructure is built, if not on a quicksand, at least in uncomfortable nearness to an abyss.

As stated above, the problem had been considered in 1774 by Laplace* whose approximation by aid of Stirling's Theorem leads us directly to the normal curve. I shall later on repeat and to some extent modify on a broader basis Laplace's investigation. But Laplace was really only following Bayes†—and for our purposes

* *Mémoires de mathématique et de physique présentés à l'Académie par divers savans*, T. VI. p. 6. Paris, 1774.

† Bayes' work was communicated after his death to the Royal Society by Price: see *Phil. Trans.* Vol. LIII. pp. 269—271, 370—375. Condorcet also gave the main result in 1783, *Histoire de l'Académie*, 1786, p. 539, Paris, 1788 and also in the "Essai sur l'application de l'analyse à la probabilité...," p. 188, Paris, 1785. Thus Condorcet wrote between the publication of Laplace's memoir of 1774 and of the *Théorie analytique des Probabilités*, Paris, 1812. Condorcet, however, supplied the combination factor omitted by Laplace. At the same time (*Essai*, p. lxxxiii) he admitted Bayes and Price's priority, while remarking that Laplace had been the first to treat it analytically. He recognised the existence of the hypothesis of "the stability of the statistical ratios," it is not equally clear that he recognised the need for "the equal distribution of ignorance." I do not think it is correct to say that Laplace was the first to treat the problem analytically. It all turns on the evaluation of the incomplete B-function. The methods of quadrature of Bayes and Price may be somewhat primitive, but I cannot see that they are much rougher than those used on this occasion by Laplace. There is no special merit in reducing any integral to terms in exponentials, unless these give an adequate approximation to the sought value. And Laplace does not really measure the closeness of his approximation nor indicate where it fails.

the fundamental principle on which Laplace's more complete solution is based may be extricated from Bayes' much more involved reasoning. None of the early writers on this topic—all approaching the subject from the mathematical theory of games of chance—seem to have had the least inkling of the enormous extension of their ideas, which would result in recent times from the application of the theory of random sampling to every phase of our knowledge and experience—economic, social, medical, and anthropological—and to all branches of observation whether astronomical, physical or psychical. Hence they must not be too severely handled if their "bag of balls" hypotheses seem too flimsy a structure for modern statistical theory.

Bayes in his original paper supposes balls rolled on a table and *equally likely to halt anywhere on the breadth of the table.* Condorcet and Laplace only generalise this in appearance by supposing all probabilities equally likely before the event. It is the same hypothesis expressed in more general words. Each repetition is supposed to be independent and the elementary algebra view of compound probability is accepted without hesitation. Thus we have two hypotheses to deal with:

(i) the hypothesis that *a priori* we ought to distribute our ignorance of the chance of a marked individual occurring *equally,*

(ii) the hypothesis that earlier occurrences do not modify the chance of later trials.

(2) The chief criticism of the theory of inverse probabilities has been based on the want of generality in the first hypothesis—what Boole has termed "the equal distribution of our ignorance" (*Laws of Thought,* p. 370, 1854) or the assigning to the appearance of a factor whose real probability is unknown to us all degrees of probability and then making these degrees all equally likely to occur. This is not the sole, but I think the chief feature of Boole's attack on the theory of inverse probabilities. Why should we, he asks, distribute our ignorance equally? Other sorts of distributions may occur, and we know do occur in chance problems. Why this assumption in the particular case? Dr Venn's strongly unfavourable criticism of inverse probabilities seems also based on objection to the principle of equal distribution of ignorance*. Strangely enough it seems to me, he is willing to admit the hypothesis when it is applied to a comparison of the probable contents of two bags of balls, while refusing to consider it valid in the case of a single bag. Thus he considers the problem: "of 10 cases treated by Lister's method seven did well and three suffered from blood-poisoning: of 14 treated with ordinary dressings nine did well and five had blood poisoning, what are the odds that the success of Lister's method was due to chance"? (p. 187) and remarks that the "bag of balls" method—i.e. the assumption of the equal prevalence of the different possible kinds of bag—seems "to be the only reasonable way of treating the problem, if it is to be considered capable of numerical solution at all." But I fail to see why the application of the hypothesis of the equal distribution of ignorance to Lister's method *and*

* *The Logic of Chance,* p. 182 *et seq.*

to ordinary dressings is logically more legitimate than to either alone. The only argument must be that in some way a *wrong* distribution of the *a priori* chance of blood poisoning is more likely to correct itself if applied for comparative than for absolute purposes. Dr Venn does not show, however, that any distribution, right or wrong, would lead in the comparative case to the same result. I cannot help fancying that had Dr Venn come across the problem as a bag of balls problem he would have rejected the equal distribution of ignorance solution. But when he meets the same problem in a vital problem of conduct, he realises that some solution is essential, there is a wide experience in man of the stability of statistical ratios, and for practical purposes he needs to apply this even to small samples of new experiences. What he demands and rightly demands is a measure of the faith he is to put—as a guide to immediate conduct—in small experience, and Bayes' hypothesis gives him a rule, which if rough, corresponds not wholly badly with his sum total of small experiences. The mathematicians in their customary manner first stated an hypothesis which simplified their analysis without questioning whether it had any foundation in experience. But Maxwell's brilliant paradox—that the theory of probabilities is the only logic for the practical man—means more than it superficially conveys. The practical man is always working by comparative probabilities even if he has not reduced his appreciations to numbers. He would certainly scoff at the idea that a first sample should not influence his judgment of what a second sample or what the bulk would be like. For such an idea would render most actions in life impossible. If science cannot measure the degree of probability involved—so much the worse for science. The practical man will stick to his appreciative methods until it does, or will accept the results of inverse probability of the Bayes-Laplace brand till better are forthcoming.

Now let us see exactly where we stand. Notwithstanding the criticisms of Boole and Venn all branches of science have adopted the theory of "probable errors": they have applied past experience of limited samples to predict what deviations are likely to occur from this past experience in future experience, and mankind acts in accordance with a firm conviction in the relative stability of statistical ratios. But any numerical appreciation of the reasonableness of this conduct is apparently based on the "equal distribution of ignorance" or ultimately on such a quaint idea as that of Bayes that his balls might roll anywhere on the table with equal probability.

Edgeworth has done all that is feasible to transfer our faith in inverse probabilities from purely mathematical to more sound observational foundations. He wrote in 1884*, "The ridicule which has been heaped upon Bayes' theorem and the inverse method will be found only applicable to the pretence here deprecated of eliciting knowledge out of ignorance, something out of nothing. The most formidable objection is that which was made by Boole and is repeated by Mr Venn, Mr Pierce and others with approbation. Our procedure in treating one value as *a priori* not less likely than another is, it is said, of a quite arbitrary character, and

* *Mind*, Vol. IX. p. 230.

apt to lead to different conclusions from the plausible one which we have reached by accident."

Edgeworth takes the bull by the horns and broadly speaking asserts that later experience has shown us that in innumerable cases where we were *a priori* ignorant the chances of given events really did not cluster. "We take our stand upon the fact that probability-constants occurring in nature present every variety of fractional value; and that natural constants in general are found to show no preference for one number rather than another. Acting on which supposition, while in particular cases we shall err, in the long run we shall find our account" (p. 231).

In other words Edgeworth returns to the appeal to experience from which Bayes and Laplace ought to have started. He asserts that in our ignorance we must appeal to our knowledge of kindred experience, and that statistical constants do not cluster in such experience. "I submit," he writes, "the assumption that any probability-constant about which we know nothing in particular is as likely to have one value as another is grounded upon the rough but solid experience that such constants do as a matter of fact, as often have one value as another" (p. 230). Unfortunately he does not give us any numerical data—except a few isolated illustrations—to prove that the universe of statistical ratios exhibits a rectangular frequency curve. Some captious critics might assert a conviction that *their* experience showed that chances ranged themselves like the horns of the bull, and that absolute non-occurrence and persistent occurrence were in the world at large the most emphasised probabilities. Our professor impaled thus upon the horns of the bull he has ventured to seize might leave the doctrine of inverse probabilities in a worse plight than he found it!—His practical safety depends on the extreme difficulty of testing by actual *numerical* experience the "fact" that natural constants in general show no preference for one number rather than another. At the best we may "feel" it is so, rather than hope to demonstrate it by forming a random sample which would cover the immensely wide fields of natural frequencies.

It has occurred to me, however, that possibly the bull itself is a chimera, and there may be no need whatever to master it. In short, is it not possible that any *continuous* distribution of *a priori* chances would lead us equally well to the Bayes-Laplace result? If this be so, then the main line of attack of its critics fails. If they then continue their assault on the ground that there are cases in which the repetitions of events are not independent, it is conceivable, that we who defend inverse probabilities as not only the basis of modern statistical theory, but also as the arithmological justification for the most ordinary actions and beliefs of the practical man can again counter our adversaries.

(3) Let us examine the problem on the lines of Bayes' original investigation into *a priori* chances. We may reduce that investigation to its simplest terms as follows: A line of length a is taken and a stroke made on it at random, and its position at distance x from one end is unknown. Afterwards n strokes are made at random on the line, and p fall in the segment 0 to x and $q = n - p$ in the segment x to a. The former are looked upon as "successes," the latter as "failures."

For a given position between x and $x+\delta x$, Bayes takes the total probability to be

$$\frac{\delta x}{a}\left(\frac{x}{a}\right)^p\left(1-\frac{x}{a}\right)^q.$$

It is against this first stage that the objection is raised that the chance of placing the first stroke is taken as $\delta x/a$, and of succeeding strokes x/a or $(a-x)/a$. All this is on the assumption that the first stroke *a priori* is equally likely to be anywhere in the line of length a, or freed from the analogy of strokes, the *a priori* chance is taken to have any value between 0 and 1 with equal likelihood. But there is really no necessity for this limitation. Suppose the frequency curve for strokes along the line to be given by any continuous function of x, say $y=\frac{\phi(x)}{a}$. Then the chance of a stroke occurring between x and $x+\delta x$ will be $\phi(x)\,\delta x/a$ instead of $\delta x/a$. Further, the chance P_x of a stroke afterwards occurring between 0 and x will be

$$P_x=\int_0^x \phi(x)\,dx/a,$$

and between x and a $\qquad Q_x=1-P_x=\int_x^a \phi(x)\,dx/a.$

Clearly $P_0=0$ and $P_a=1$. Thus the probability of the combined event, since

$$\delta P_x=\phi(x)\,\delta x/a,$$

will be $$\delta P_x P_x^p\, Q_x^q\,\frac{(p+q)!}{p!\,q!}.$$

Proceeding as in Bayes' or Laplace's manner, we have for the probability that the unknown original probability lies between P_b and P_c (i.e. x between b and c)

$$\frac{\int_{P_b}^{P_c} dP_x P_x^p (1-P_x)^q}{\int_0^1 dP_x P_x^p (1-P_x)^q}.$$

Or, again, for the whole chance that there will be in a further m trials, r successes and s failures,

$$C_r=\frac{\int_0^1 dP_x P_x^{p+r}(1-P_x)^{q+s}}{\int_0^1 dP_x P_x^p (1-P_x)^q}\times\frac{(r+s)!}{r!\,s!}.$$

Now it is clear that the above integrals will take the same values

$$B(p+r+1,\,q+s+1) \quad\text{and}\quad B(p+1,\,q+1)$$

whether we replace P_x by x or not. But if we replace P_x by x, we have exactly the formula reached by Laplace on the basis of "the equal distribution of ignorance," to use Boole's phrase. Thus it would appear that the fundamental formula of Laplace*, i.e.

$$C_r=\frac{B(p+r+1,\,q+s+1)}{B(p+1,\,q+1)\,B(r+1,\,s+1)},$$

* Including the $B(r+1,\,s+1)$ term omitted by Laplace.

in no way depends on the equal distribution of ignorance. It is sufficient to assume any continuous distribution—which may vary from one type of *a priori* probability problem to a second—in order to reach the basis of the fundamental problem in practical statistics, i.e. the probability that in a second trial of m events following a p, q experience, r will be successful and s fail.

(4) I now propose to develop the above expression rather more completely than Laplace has done. We shall first replace the B-functions by Γ-functions. Then we shall use Stirling's theorem on the assumption that none of the quantities p, q, r or s are small. As we solely need to find the variation of C_r with r, we need not trouble about terms not involving r or s. We have

$$C_r = \frac{1}{B(p+1, q+1)} \frac{\Gamma(p+r+1) \cdot \Gamma(q+s+1) \cdot \Gamma(r+s+2)}{\Gamma(p+q+r+s+2)\,\Gamma(r+1)\,\Gamma(s+1)}$$

$$= \frac{e^{-n}}{B(p+1, q+1)} \frac{\Gamma(m+2)}{\Gamma(n+m+2)} \frac{(p+r)^{p+r+\frac{1}{2}}\,(q+s)^{q+s+\frac{1}{2}}}{r^{r+\frac{1}{2}}\,s^{s+\frac{1}{2}}} \times \chi,$$

where χ

$$= \frac{\left(1 + \frac{1}{12(p+r)} + \frac{1}{288(p+r)^2} + \ldots\right)\left(1 + \frac{1}{12(q+s)} + \frac{1}{288(q+s)^2} + \ldots\right)}{\left(1 + \frac{1}{12r} + \frac{1}{288r^2} + \ldots\right)\left(1 + \frac{1}{12s} + \frac{1}{288s^2} + \ldots\right)}$$

We now take $$r = \frac{m}{n}p + h, \quad s = \frac{m}{n}q - h,$$

and put very approximately

$$\chi = e^{\frac{1}{12(p+r)} + \frac{1}{12(q+s)} - \frac{1}{12r} - \frac{1}{12s}}$$

We may write

$$C_r = \left\{\frac{e^{-n}p^p q^q\,\Gamma(m+2)}{B(p+1, q+1)\,\Gamma(n+m+2)}\right\} \frac{\left(1+\frac{r}{p}\right)^{p+r+\frac{1}{2}}\left(1+\frac{s}{q}\right)^{q+s+\frac{1}{2}}}{\left(\frac{r}{p}\right)^{r+\frac{1}{2}}\left(\frac{s}{q}\right)^{s+\frac{1}{2}}}\chi.$$

The factor in large curled brackets we will call C; it does not vary with r. The second factor, involving $\frac{r}{p}$ and $\frac{s}{q}$, we will call u, and the remainder is χ. We have

$$\log_e C_r = \log_e C + \log_e u + \log_e \chi,$$

and we need the development only of the last two logarithms. We shall write $(n+m)/n = 1+\rho$, and $p/n = P$, $q/n = Q$ will give the chances of success and failure as estimated on the first sample. We have

$$\log_e u = \{p(1+\rho) + h + \tfrac{1}{2}\} \log\left(1 + \rho + \frac{h}{p}\right) + \{q(1+\rho) - h + \tfrac{1}{2}\}\left(\log 1 + \rho - \frac{h}{q}\right)$$

$$- \{\rho p + h + \tfrac{1}{2}\} \log\left(\rho + \frac{h}{p}\right) - \{\rho q - h + \tfrac{1}{2}\} \log\left(\rho - \frac{h}{q}\right).$$

Expanding and collecting terms in powers of h, we have

$$\log_e u = \{(p+q)(1+\rho)+1\}\log_e(1+\rho) - \{(p+q)\rho+1\}\log_e\rho - \tfrac{1}{2}h\frac{1}{\rho(1+\rho)}\frac{q-p}{pq}$$
$$-\tfrac{1}{2}h^2\left(\frac{1}{\rho(1+\rho)}\frac{p+q}{pq} - \frac{1+2\rho}{2\rho^2(1+\rho)^2}\frac{p^2+q^2}{p^2q^2}\right)$$
$$+\tfrac{1}{6}h^3\left\{\frac{1+2\rho}{\rho^2(1+\rho)^2}\left(\frac{1}{p^2}-\frac{1}{q^2}\right) - \frac{1+3\rho+3\rho^2}{\rho^3(1+\rho)^3}\left(\frac{1}{p^3}-\frac{1}{q^3}\right)\right\} + \text{etc.},$$
$$= \{n(1+\rho)+1\}\log_e(1+\rho) - \{n\rho+1\}\log_e\rho - \tfrac{1}{2}h\frac{1}{m(1+\rho)}\frac{Q-P}{PQ}$$
$$+\tfrac{1}{2}h^2\left(\frac{1}{m(1+\rho)PQ} - \frac{1+2\rho}{2m^2(1+\rho)^2}\frac{1-2PQ}{P^2Q^2}\right)$$
$$+\tfrac{1}{6}h^3\left(\frac{1+2\rho}{m^2(1+\rho)^2}\frac{Q-P}{P^2Q^2} - \frac{1+3\rho+3\rho^2}{m^3(1+\rho)^3}\frac{Q^3-P^3}{P^3Q^3}\right) + \text{etc.}$$

Hence, putting $\sigma^2 = m(1+\rho)PQ$,

$$\log_e u = \{n(1+\rho)+1\}\log_e(1+\rho) - (n\rho+1)\log_e\rho - \frac{1}{2}\frac{h}{\sigma}\frac{Q-P}{\sqrt{m}\sqrt{1+\rho}\sqrt{PQ}}$$
$$-\frac{1}{2}\frac{h^2}{\sigma^2}\left(1-\frac{1+2\rho}{2m(1+\rho)}\frac{1-2PQ}{PQ}\right)$$
$$+\frac{1}{6}\frac{h^3}{\sigma^3}\left(\frac{1+2\rho}{\sqrt{m}\sqrt{1+\rho}}\frac{Q-P}{\sqrt{PQ}} - \frac{1+3\rho+3\rho^2}{m^{\frac{3}{2}}(1+\rho)^{\frac{3}{2}}}\frac{(Q-P)(1-QP)}{(PQ)^{\frac{3}{2}}}\right) + \text{etc.}$$

Similarly

$$\log_e\chi = -\frac{1}{12PQ}\frac{1}{m(1+\rho)} - \frac{1}{12}\frac{h}{\sigma}\frac{1+2\rho}{m^{\frac{3}{2}}(1+\rho)^{\frac{3}{2}}}\frac{P-Q}{(PQ)^{\frac{3}{2}}}$$
$$-\frac{h^2}{12\sigma^2}\frac{1+3\rho+3\rho^2}{m^2(1+\rho^2)}\frac{1-3PQ}{PQ} - \text{etc.}$$

If we stop at terms in $\frac{1}{m}$ we see that $\log_e\chi$ contributes only to the constant term and this may be included in C. Into the same factor C may be put the logarithmic terms in $\log_e u$. Thus finally we have*:

$$C_r = Ce^{-\frac{1}{2}\frac{h^2}{\sigma^2}\left(1-\frac{1+2\rho}{2m(1+\rho)}\frac{1-2PQ}{PQ}\right)}$$
$$\times e^{-\frac{1}{2}\frac{h}{\sigma}\frac{Q-P}{\sqrt{m}\sqrt{1+\rho}\sqrt{PQ}} + \frac{1}{6}\frac{h^3}{\sigma^3}\frac{1+2\rho}{\sqrt{m}\sqrt{1+\rho}}\frac{Q-P}{\sqrt{PQ}}}.$$

We note the following facts with regard to this result:

(i) The series converges with a factor $1/\sqrt{m}$. Unless this be small we cannot neglect the terms in h and h^3. In other words deviations in excess and defect of mP are *not* equally probable. Thus skew frequency rather than the Gaussian hypothesis is indicated from the very start of the investigation if our second sample be not considerable.

* The existence of the term in $(h/\sigma)^3$ of the same order as that in h/σ prevents us treating the result as a normal curve with shifted centre.

(ii) Examining the odd terms we see that within the range

$$h = \pm \frac{3}{1+2\rho}\sigma = \pm \frac{3n}{n+2m}\sigma$$

positive deviations are less frequent than negative deviations, but beyond this range the positive are more frequent than the negative. In other words *large* excesses are more probable than large defects, but small excesses are less probable than small defects from the result of our first sample.

(iii) It is not legitimate to keep the term in h^2 in $\frac{1}{m}$, for this is really the order of the lowest term in h^4/σ^4. Without including that term our degree of approximation will only be to $1/\sqrt{m}$.

(iv) For the very narrow case of $Q = P$, or equal chance of success and failure, of course the odd terms vanish.

If we include in this case the lowest term in h^4/σ^4 we have

$$C_r = Ce^{-\frac{1}{2}\frac{h^2}{\sigma^2}\left(1-\frac{1+2\rho}{m(1+\rho)}\right)-\frac{1}{12}\frac{h^4}{\sigma^4}\frac{1+3\rho+3\rho^2}{m(1+\rho)}}.$$

The case of $P = Q$ is, however, of such secondary importance that this second approximation on Laplace's lines is of little significance*.

(v) If our second sample is so large that terms in $\frac{1}{\sqrt{m}}$ are negligible, then our frequency becomes

$$C_r = Ce^{-\frac{1}{2}\frac{h^2}{\sigma^2}}$$

This result was reached by Laplace before Gauss. Neither stated the very narrow limitations of the formula.

Besides the condition that m is to be very large, we must also note that neither Q nor P can be very small. We obtain quite different results, if we suppose m large and P small so that mP remains finite.

(vi) Thus the Gauss-Laplacian distribution fails:

(*a*) for small samples. Its whole method of deduction is then wrong for Stirling's Theorem is invalid;

(*b*) when the sample is large, but the probability of occurrence is small, so that mP is finite and small.

The whole of this subject has now-a-days been reconsidered under the topics of "Small Samples" and the "Law of Small Numbers."

* A method of investigating the frequency constants of such a distribution has recently been given by Forsyth, *Messenger of Mathematics*, Vol. XLVIII. pp. 131–44.

Laplace's method gives the "normal curve" as an approximation to the original series, when we make very limiting assumptions as to n, m, P. It led to his suggestion that $\int e^{-\frac{1}{2}x^2}dx$ should be tabulated, and eventually to the computation of the so-called "probability integral."

(5) It will be seen that Laplace's method is only a rough method of approximation to our original series for C_r. That series is only a special case of the general hypergeometrical series, and we have at once thrust upon us the problem: Can no better curve be found than the Gauss-Laplacian $y = y_0 e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$ to represent the sum of any number of terms of the hypergeometrical series? The answer is that curves infinitely better which demand no restriction on the values of m, n, P can be determined, and these curves can with just as much validity be called probability curves as the above normal curve; and the integrals of their areas up to any given value have equal claim to be "probability integrals."

Let us return to Laplace's starting-point on p. 6. We have at once for the ratio:

$$\frac{C_{r+1}}{C_r} = \frac{(p+r+1)\,s}{(r+1)\,(q+s)} = \frac{(p+r+1)\,(m-r)}{(r+1)\,(q+m-r)}.$$

Hence
$$\frac{C_{r+1}-C_r}{C_{r+1}+C_r} = \frac{p\,(m+1)-n-rn}{m\,(p+2)+q+r\,\{2\,(m-1)-p+q\}-2r^2}.$$

Divide by n and put $p/n = P$, $q/n = Q$ as before, and we have

$$\frac{C_{r+1}-C_r}{C_{r+1}+C_r} = \frac{P\,(m+1)-1-r}{m\left(P+\frac{2}{n}\right)+Q+r\left(Q-P+2\,\frac{m-1}{n}\right)-\frac{2}{n}r^2}.$$

Now suppose C_0, $C_1 \ldots C_r$, $C_{r+1} \ldots$ to be plotted as a histogram, i.e. as a series of rectangles of base c and heights C_0/c, $C_1/c \ldots C_r/c$, $C_{r+1}/c \ldots$. Let the tops of the midordinates of these rectangles be joined, so as to form a polygonal figure P_0, P_1, $P_2 \ldots P_r$, $P_{r+1} \ldots$. If we take the origin at the ordinate C_0/c we shall have for abscissae of the angles $X_r = rc$, $X_{r+1} = r\pi c \ldots$ and for ordinates $y_r = C_r/c$, $y_{r+1} = C_{r+1}/c$. Thus the abscissa and the ordinate of the midpoint of a side of the polygon will be $x_{r+\frac{1}{2}} = (r+\frac{1}{2})\,c$ and $y_{r+\frac{1}{2}} = \frac{1}{2}\,(C_{r+1}+C_r)/c$. Accordingly we have, if $\Delta x = c$:

$$\frac{1}{y_{r+\frac{1}{2}}}\,\frac{\Delta y_{r+\frac{1}{2}}}{\Delta x}$$
$$= \frac{2}{c}\,\frac{c\,\{P\,(m+1)-\frac{1}{2}\}-X_{r+\frac{1}{2}}}{c\left\{m\left(P+\frac{2}{n}\right)+Q\right\}+c\left\{(Q-P)+\frac{2\,(m-1)}{n}\right\}(X_{r+\frac{1}{2}}-\frac{1}{2}c)-\frac{2}{n}(X_{r+\frac{1}{2}}-\frac{1}{2}c)^2}$$

Now let us transfer the origin to the mode, i.e. take

$$-x_{r+\frac{1}{2}} = c\,\{P\,(m+1)-\tfrac{1}{2}\}-X_{r+\frac{1}{2}}.$$

Hence after some reductions we find

$$\frac{1}{y_{r+\frac{1}{2}}}\frac{\Delta y_{r+\frac{1}{2}}}{\Delta x}=\frac{-x_{r+\frac{1}{2}}}{PQ(m+1)\left(1+\frac{m+1}{n}\right)c^2+c\left\{(Q-P)\left(\frac{1}{2}+\frac{m+1}{n}\right)\right\}x_{r+\frac{1}{2}}-\frac{1}{n}x^2{}_{r+\frac{1}{2}}}$$

Take $$\sigma_0^2=PQ(m+1)\left(1+\frac{m+1}{n}\right)c^2,$$

and we may write

$$\frac{\text{Slope of side of polygon } P_rP_{r+1}}{\text{Ordinate of midpoint of side}}=-\frac{x_{r+\frac{1}{2}}}{\sigma_0^2}\frac{1}{1+\dfrac{(Q-P)\left(\frac{1}{2}+\frac{m+1}{n}\right)}{\sqrt{QP(m+1)\left(1+\frac{m+1}{n}\right)}}\dfrac{x_{r+\frac{1}{2}}}{\sigma_0}-\dfrac{1}{n}\dfrac{x^2{}_{r+\frac{1}{2}}}{\sigma_0^2}}$$

Now Laplace's hypotheses amount to taking both n and m very large and n relatively large as compared with m. Thus we have, if we neglect the second and third term in the denominator,

$$\frac{\text{Slope of } P_rP_{r+1}}{\text{Ordinate}}=-\frac{x_{r+\frac{1}{2}}}{\sigma_0^2}.$$

The same result would be reached, if we took $Q=P$, i.e. took equal chances for each contributory event—which is really Gauss' equality of chance for errors in excess and defect—together with n the number of cause groups in the first sample very large.

To get the Gaussian or normal curve we must then replace differences by differentials and we have

$$\frac{1}{y}\frac{dy}{dx}=-\frac{x}{\sigma_0^2}.$$

Laplace, however, fails to introduce the condition that neither P nor Q is to be small which is essential.

Now there is a point here of considerable importance. We do *not* assume that differences may be replaced by differentials. What we have reached is a fundamental geometrical property of the hypergeometrical histogram and what we do is to seek for a continuous curve, which possesses this same geometrical property.

If our first sample be very large, i.e. we have exact knowledge of the bag or population contents, the last term in $\frac{1}{n}$ will vanish and we have

$$\frac{1}{y}\frac{dy}{dx}=\frac{-x}{\sigma_0^2+\frac{1}{2}c(Q-P)x},$$

where $$\sigma_0^2=PQ(m+1)c^2.$$

This is the curve which corresponds as closely to the skew binomial histogram as the normal curve has been shown by Laplace to correspond to the symmetrical binomial histogram, or to the skew binomial even if n be very large and the bases of the histogram very small.

In general however when none of these conditions hold we have

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{a_0 + a_1 x + a_2 x^2},$$

where $a_0 = PQ(m+1)\left(1+\frac{m+1}{n}\right)c^2$ and is *positive*,

$a_1 = (Q-P)\left(\frac{1}{2}+\frac{m+1}{n}\right)c$ and may be of either sign,

$a_2 = -\frac{1}{n}$ and is *negative*.

Thus for this particular case we may throw the result into the form

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{b_0(b_1-x)(b_2+x)} = \frac{1}{b_0(b_1+b_2)}\left\{\frac{b_2}{b_2+x} - \frac{b_1}{b_1-x}\right\}.$$

Hence

$$\log_e y = \text{constant} + \frac{1}{b_0(b_1+b_2)}\{b_2\log_e(b_2+x) + b_1\log_e(b_1-x)\}.$$

Let $$s_1 = \frac{b_1}{b_0(b_1+b_2)}, \quad s_2 = \frac{b_2}{b_0(b_1+b_2)}.$$

Thus $$y = y_0\left(1+\frac{x}{b_2}\right)^{s_2}\left(1-\frac{x}{b_1}\right)^{s_1},$$

where y_0 is the modal ordinate.

This is a limited range curve, of which the partial area is expressible in terms of the incomplete B-function. In other words, the so-called "probability integral" is only a very special case of the incomplete B-function, which is in this sense the general probability integral.

We easily find

$$b = b_1 + b_2 = c\{\tfrac{1}{2}n + m + 1 - n\sqrt{PQ}\}^{\frac{1}{2}}\{\tfrac{1}{2}n + m + 1 + n\sqrt{PQ}\}^{\frac{1}{2}},$$

and $$b_1 - b_2 = c(Q-P)(\tfrac{1}{2}n + m + 1).$$

Thus $$b_1 = \tfrac{1}{2}b + \tfrac{1}{2}c(Q-P)(\tfrac{1}{2}n + m + 1),$$

$$b_2 = \tfrac{1}{2}b - \tfrac{1}{2}c(Q-P)(\tfrac{1}{2}n + m + 1).$$

Hence $s_1 = nb_1/b$

$$= \tfrac{1}{2}n\left(1 + \frac{(Q-P)(\tfrac{1}{2}n+m+1)}{\{\tfrac{1}{2}n+m+1-n\sqrt{PQ}\}^{\frac{1}{2}}\{\tfrac{1}{2}n+m+1+n\sqrt{PQ}\}^{\frac{1}{2}}}\right).$$

Similarly $$s_2 = \tfrac{1}{2}n\left(1 - \frac{(Q-P)(\tfrac{1}{2}n+m+1)}{\{\tfrac{1}{2}n+m+1-n\sqrt{PQ}\}^{\frac{1}{2}}\{\tfrac{1}{2}n+m+1+n\sqrt{PQ}\}^{\frac{1}{2}}}\right).$$

Or, writing $$\frac{1}{2}+\frac{m+1}{n}=\epsilon \text{ for brevity,}$$

$$s_1=\tfrac{1}{2}n\left(1+\frac{(Q-P)\,\epsilon}{(\epsilon^2-PQ)^{\frac{1}{2}}}\right),$$

$$s_2=\tfrac{1}{2}n\left(1-\frac{(Q-P)\,\epsilon}{(\epsilon^2-PQ)^{\frac{1}{2}}}\right),$$

$$b=cn\,(\epsilon^2-PQ)^{\frac{1}{2}}.$$

$$b_1=\tfrac{1}{2}\{b+nc\,(Q-P)\,\epsilon\},$$

$$b_2=\tfrac{1}{2}\{b-nc\,(Q-P)\,\epsilon\}.$$

Distance of mode from start of hypergeometrical

$$=c\,\{P\,(m+1)-\tfrac{1}{2}\}.$$

Further distance from mode to mean

$$=(b_1-b_2)/(n+2)=c\,(Q-P)\left(\frac{1}{2}+\frac{m}{n+2}\right).$$

(6) *Illustration.* I owe an illustrative example of this to Mr E. C. Rhodes, and the accompanying diagram to Miss A. Davin. Suppose 20% of individuals in a sample of 1000 have been found to possess a given character, what will be the chances of such percentage arising in a further sample of 100 individuals?

This is the type of problem which arises every day in statistical work and which I term the *fundamental* problem of statistics. Only too often the first sample is treated as indefinitely great, and the probabilities calculated from the probability integral of the normal curve on the hypothesis that $\sigma_0=\sqrt{mPQ}\,.\,c$. In Diagram I the results for the normal curve have been bettered by taking

$$\sigma_0=\sqrt{(m+1)\,PQ\left(1+\frac{m+1}{n}\right)}\,c.$$

It is still quite inadequate to provide the requisite percentages. These are plotted as the small full circles, and they are seen to lie most closely on the skew curve reckoned with the above constants. The integral of the curve was then obtained by the integraph, and the areas read off to the nearest tenth. The results are given in Table I, p. 13. See Diagram I.

It will be seen that the skew curve gives frequencies never differing more than a fraction per cent. from the series, while the deviations of the Gaussian are much larger. A still better result would have been obtained had we fitted the series by moments rather than by the geometrical relation. This will be illustrated in the next example.

(7) It may be of interest to see what does really happen when we suppose n infinite or the population "known."

Returning to p. 10 we have

$$\sigma_0^2=PQ\,(m+1)\,c^2,$$

TABLE I.

Percentages of Frequency from Mean to Deviations in First Column.

	In Excess of Mean				In Defect of Mean		
		Calculated				Calculated	
Up to	Actual	Gaussian	Skew Curve	Up to	Actual	Gaussian	Skew Curve
·5	4·73	4·74	4·73	− ·5	4·73	4·74	4·72
1·5	13·77	13·97	13·77	− 1·5	14·08	13·97	14·02
2·5	21·98	22·44	22·06	− 2·5	22·80	22·44	22·76
3·5	29·04	29·79	29·27	− 3·5	30·46	29·79	30·42
4·5	34·83	35·83	35·07	− 4·5	36·78	35·83	36·72
5·5	39·40	40·51	39·62	− 5·5	41·65	40·51	41·50
6·5	42·81	43·94	43·09	− 6·5	45·15	43·94	44·99
7·5	45·23	46·31	45·41	− 7·5	47·49	46·31	47·28
8·5	46·88	47·86	47·12	− 8·5	48·93	47·86	48·69
9·5	47·96	48·82	48·21	− 9·5	49·74	48·82	49·46
10·5	48·63	49·38	48·89	− 10·5	50·16	49·38	49·91
11·5	49·04	49·69	49·28	− 11·5	50·37	49·69	50·11
12·5	49·26	49·86	49·50	− 12·5	50·46	49·86	50·18
13·5	49·39	49·94	49·66	− 13·5	50·49	49·94	50·18
14·5	49·45	49·97	49·72	− 14·5	50·49	49·97	50·18
15·5	49·49	49·99	49·77	− 15·5	—	49·99	—
16·5	49·51	50·00	49·82	− 16·5	—	50·00	—

and
$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{\sigma_0^2\left(1+\frac{\frac{1}{2}(Q-P)}{\sqrt{QP(m+1)}}\frac{x}{\sigma_0}\right)}$$

will be the equation to the curve associated now with the binomial $(p+q)^m$. Let us write it in the form

$$\frac{1}{y}\frac{dy}{d\left(\frac{x}{a}\right)} = \frac{-\lambda\frac{x}{a}}{1+\frac{x}{a}}.$$

Here $$a = \frac{2\sqrt{QP(m+1)}}{Q-P}\,\sigma_0 = \frac{2PQ(m+1)}{Q-P}\,c, \text{ and } \lambda = \frac{4QP(m+1)}{(Q-P)^2},$$

and integrating $$y = y_0 e^{-\lambda\frac{x}{a}}\left(1+\frac{x}{a}\right)^{\lambda},$$

where the origin is the mode.

The distance from the mode to the mean is

$$d = a/\lambda = \frac{1}{2}\frac{Q-P}{\sqrt{QP(m+1)}}\,\sigma_0 = \tfrac{1}{2}(Q-P)\,c$$

and
$$y_0 = \frac{M}{a}\frac{\lambda^{\lambda+1}e^{-\lambda}}{\Gamma(\lambda+1)}.$$

Now the true standard deviation of the *curve* is not σ_0*, but is given by

$$\sigma = \frac{\sqrt{\lambda+1}}{\lambda}\, a,$$

or
$$\sigma^2 = \{QP(m+1) + \tfrac{1}{4}(Q-P)^2\}\, c^2 = \sigma_0^2 + \tfrac{1}{4}(Q-P)^2 c^2$$
$$= \sigma_0^2 \text{ nearly,}$$

if m be fairly large and neither Q nor P small.

In both cases the (standard deviation)^2 of the curve exceeds that of the series by a term depending on the square of the distance between mean and mode. Unless either P or Q are small, this will be a small difference, a fraction of the square of the plotting unit c. As the curves are not reached by equating moments, but from a geometrical property common to the polygon of frequency and to the curve, this deviation is to be anticipated. It is the more to be anticipated as in the case of the curve we are dealing with the moments of continuous areas, and in the case of the series with concentrated lumps†.

* Similarly for the limited range curve of p. 11:

$$\sigma_0^2 = (m+1)\,PQ\left(1+\frac{m+1}{n}\right)c^2 = nPQ\,(\epsilon^2 - \tfrac{1}{4})\,c^2.$$

But the true standard deviation σ is given by

$$\sigma^2 = \frac{b^2(s_1s_2+s_1+s_2+1)}{(s_1+s_2+2)^2(s_1+s_2+3)} = \frac{c^2}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\left\{nPQ\,(\epsilon^2-\tfrac{1}{4}) + \left(1+\frac{1}{n}\right)(\epsilon^2 - PQ)\right\}$$

$$= \frac{1}{1+\frac{3}{n}}\,\sigma_0^2 + \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\,\epsilon^2(P-Q)^2c^2$$

$$= \frac{1}{1+\frac{3}{n}}\,\sigma_0^2 + \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\,d^2,$$

where d = distance from mean to mode.

† The true (standard deviation)^2 of the binomial is $\sigma_1^2 = mPQc^2$. Thus

$$\sigma^2 = \{mPQ + PQ + \tfrac{1}{4}(Q-P)^2\}\,c^2 = \sigma_1^2 + \tfrac{1}{4}c^2,$$

or the (standard deviation)^2 of the binomial series is less than that of curve by $\frac{1}{4}c^2$.

The true standard deviation of the series when n is not indefinitely great is given by

$$\sigma_2^2 = c^2\left[\frac{mPQ\left(1+\frac{m+2}{n}\right)}{1+\frac{3}{n}} + \frac{\frac{m}{n}\left(1+\frac{1}{n}\right)\left(1+\frac{m+2}{n}\right)}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}(P-Q)^2\right]$$

$$= \frac{1}{1+\frac{3}{n}}\,\sigma_0^2 + \frac{\left(1+\frac{1}{n}\right)\epsilon^2(P-Q)^2}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\,c^2 - \tfrac{1}{4}c^2\,\frac{1+\frac{1}{n}}{1+\frac{3}{n}}$$

$$= \sigma^2 - \tfrac{1}{4}c^2\,\frac{1+\frac{1}{n}}{1+\frac{3}{n}}. \quad \text{(See preceding footnote.)}$$

Thus the (standard deviation)^2 of the series is less than that of the curve by a term of the order $\frac{1}{4}c^2$.

(8) I am able to illustrate by another example kindly worked by Mr E. C. Rhodes, the accuracy of this binomial curve. He has considered the case of an indefinitely large population containing 10 % of a given character, and worked out the Type III curve by both the geometrical relationship and the equality of moments to determine the frequency distribution of samples.

TABLE II.

Table of Percentage Frequencies up to given Distances from the Mean.

In Excess of Mean					In Defect of Mean				
Up to	Gaussian	Actual	*B** Curve	*C*† Curve	Up to	Gaussian	Actual	*B* Curve	*C* Curve
·5	9·40	9·25	8·84	9·05	− ·5	9·40	9·25	9·29	9·61
1·5	26·24	24·66	24·01	24·52	− 1·5	26·24	27·34	26·76	27·55
2·5	38·30	35·42	34·71	35·40	− 2·5	38·30	41·19	40·25	41·09
3·5	45·22	41·85	41·39	41·96	− 3·5	45·22	48·99	48·03	48·57
4·5	48·40	45·18	44·89	45·35	− 4·5	48·40	51·85	51·13	51·40
5·5	49·57	46·70	46·68	46·72	− 5·5	49·57	52·37	52·00	52·10
6·5	49·91	47·31	47·38	47·53	− 6·5	49·91	52·37	52·17	52·27
7·5	49·99	47·53	47·72	47·73	− 7·5	49·99	52·37	52·17	52·27
8·5	50·00	47·61	47·78	47·73	− 8·5	50·00	52·37	52·17	52·27
9·5	50·00	47·63	47·80	47·73	− 9·5	50·00	52·37	52·17	52·27
10·5	50·00	47·63	47·83	47·73	−10·5	50·00	52·37	52·17	52·27

* *B* is the Type III curve determined by the geometrical relation.
† *C* is the Type III curve determined by equality of moments.

It will be seen that the use of moments gives a better result than the geometrical property. But Type III curve obtained either way is far closer than the Gaussian to the actual distribution. See Diagram II.

(9) It would appear, therefore, that starting from an enlarged view of Bayes' Theorem, and approaching the matter from the standpoint of Laplace, we reach, both theoretically and of course in any case of actual numbers, a better result than the normal curve by using Type I or Type III for our hypergeometrical series, i.e. by using tables of the incomplete *B*- or Γ-functions rather than the table of the probability integral (areas of the Gaussian or normal curve). It must be insisted upon that Laplace reached the probability integral as an approximation to the sum of terms in a certain hypergeometrical series, and any better approximation to the sum of those terms has greater logical validity than Laplace's integral. Other deductions of the probability integral are, so to speak, after-thoughts. Its essential origin lies in Laplace's endeavour to solve the fundamental problem in statistics. The so-called Gaussian or normal curve was first introduced into statistics as a *rough and ready* solution for the sum of a certain number of terms in a definite hypergeometrical series, and the sacrosanct character of the "probability integral" and the "probable error" in the eyes of many physicists and astronomers is of the character of a dogma; it is based on authority rather than reason.

(10) It will be seen from this paper that the next stage forward in the development of the probability integral must be (i) the issue of a table of the incomplete Γ-function. Such a table is now completed and arrangements are being made for its publication; (ii) the issue of a table of the incomplete *B*-function. Such a table presents grave disadvantages. It must be a table of threefold entry and this presents more serious difficulties than even those we have encountered in the table of the incomplete Γ-function, because (*a*) of the great extent of the computations requisite and (*b*) the great cost involved in publication even if completed. I should like to draw the attention of pure mathematicians to the following problem: Can the incomplete *B*-function be expressed even approximately in terms of a limited number of incomplete Γ-functions?

Mr H. E. Soper, Miss M. Seegar and I have tried many solutions of this problem. None as yet with satisfaction to ourselves. It will not be easily solved, nor by the obvious methods of attack. Yet I am convinced that a solution is possible, if pure mathematicians of greater knowledge and larger inspiration will only deal with the matter.

I have to acknowledge my great indebtedness to Mr E. C. Rhodes for numerical work and Miss A. Davin for graphical work on the illustrations of this paper, and for help in mechanical integration of the frequency curves. Such integration of course cannot be correct to the last figure.

CORRELATION OF CEPHALIC MEASUREMENTS IN EGYPTIAN BORN NATIVES.

By MYER M. ORENSTEEN, F.S.S.

(*Ministry of Public Works, Cairo, Egypt.*)

1. *Object of the Paper.* A previous paper on the "Correlation of Anthropometrical Measurements in Cairo Born Natives*" pointed out that the coefficients of correlation resulting from the length and the breadth of the head vary much from one race to another.

The significance of this variability is suggestive and the writer accordingly undertook another investigation.

The present paper is intended to exhibit fresh coefficients of correlation for the cephalic measurements and is therefore a help to the study of the variation in question. The analysis into the variability will be confined to a separate paper already in hand.

2. *The Material.* Full particulars as to the nature of the raw material have been given in the above-mentioned paper. Suffice to repeat here that the statistics were drawn absolutely at random without any special selection whatever and that they are fairly representative of the ordinary community which they stand for. All persons measured are adult males, i.e. individuals of 20 years of age and above, and each has been classified according to his place of birth. The measurements were taken during the period 1901 to 1906.

3. *The Computations.* The total number of individuals involved in the computations is 9430. Considering the large amount of the data available, the observations have been grouped into frequencies where a standard unit of 3 millimetres was adopted.

This grouping was carried out throughout the statistics with the result that at least a dozen sub-groups was obtained for each character in each correlation-table.

It will be seen that the frequency curves have high contact at both ends, therefore Sheppard's adjustments were applied to the moments before calculating the standard deviations.

All constants have been computed twice and independently by removing the base-group by a unit of grouping.

4. *The Constants.* Table I is self explanatory. N is the number of individuals entering in each classification, A and σ are the mean and corresponding standard deviations of the group, and E_A and E_σ their respective probable errors.

* *Biometrika*, Vol. XI. Nos. 1 and 2, November 1915.

TABLE I. *Means and Standard Deviations for the Cephalic Measurements.*

Group	N	Length of the Head		Breadth of the Head	
		$A \pm E_A$	$\sigma \pm E_\sigma$	$A \pm E_A$	$\sigma \pm E_\sigma$
Alexandria Governorate	643	189·77 ± 0·16	5·95 ± 0·11	144·64 ± 0·13	4·97 ± 0·09
Cairo „	802	190·52 ± 0·14	5·90 ± 0·10	144·45 ± 0·11	4·67 ± 0·08
Canal „	127	190·56 ± 0·34	5·78 ± 0·24	145·13 ± 0·34	5·66 ± 0·24
Beheira Province	526	191·17 ± 0·17	5·81 ± 0·12	143·69 ± 0·14	4·73 ± 0·10
Gharbîya „	1104	190·90 ± 0·12	5·93 ± 0·09	143·53 ± 0·10	4·79 ± 0·07
Minufîya „	717	191·06 ± 0·15	6·00 ± 0·11	143·59 ± 0·12	4·60 ± 0·08
Daqahlîya „	504	190·39 ± 0·20	6·67 ± 0·14	143·96 ± 0·15	4·83 ± 0·10
Sharqîya „	515	190·75 ± 0·18	6·16 ± 0·13	143·65 ± 0·14	4·82 ± 0·10
Qalyubîya „	295	190·82 ± 0·23	5·87 ± 0·16	143·69 ± 0·19	4·78 ± 0·13
Gîza „	326	191·79 ± 0·22	5·78 ± 0·15	143·17 ± 0·17	4·60 ± 0·12
Fayûm „	413	191·20 ± 0·20	5·86 ± 0·14	141·84 ± 0·16	4·82 ± 0·11
Beni Sueif „	384	191·74 ± 0·19	5·65 ± 0·14	142·68 ± 0·17	4·90 ± 0·12
Minya „	491	191·72 ± 0·17	5·65 ± 0·12	142·61 ± 0·14	4·56 ± 0·10
Asyût „	887	190·91 ± 0·13	5·85 ± 0·09	142·52 ± 0·10	4·46 ± 0·07
Girga „	610	191·49 ± 0·16	6·05 ± 0·12	142·30 ± 0·12	4·41 ± 0·08
Qina „	824	191·23 ± 0·14	5·99 ± 0·10	142·30 ± 0·11	4·77 ± 0·08
Aswân „	262	190·50 ± 0·25	5·97 ± 0·18	144·26 ± 0·22	5·23 ± 0·15

It might be remarked that the means and the standard deviations given here are not exactly the same as those given by Mr J. I. Craig*. In both cases the mathematics have been carefully checked and found correct. The difference may be accounted for in the difference in the unit of the grouping method. In the former a standard unit of 3 millimetres was adopted giving at least a dozen sub-groups, whereas in the latter case the unit of grouping was invariably 2 or 3 millimetres giving about 20 sub-groups.

The difference however is insignificant (on the average three-hundredths of a millimetre) and in no case does it reach the corresponding probable error.

5. *The Coefficients of Correlation.* The following coefficients of correlations have been computed and are given in order of magnitude:

Beni Sueif Province	$r = +0{\cdot}208 \pm 0{\cdot}033$
Daqahlîya „	$r = +0{\cdot}212 \pm 0{\cdot}029$
Beheira „	$r = +0{\cdot}213 \pm 0{\cdot}028$
Minya „	$r = +0{\cdot}227 \pm 0{\cdot}029$
Girga „	$r = +0{\cdot}228 \pm 0{\cdot}026$
Minufîya „	$r = +0{\cdot}230 \pm 0{\cdot}024$
Alexandria Governorate	$r = +0{\cdot}244 \pm 0{\cdot}025$
Cairo „	$r = +0{\cdot}244 \pm 0{\cdot}025$
Sharqîya Province	$r = +0{\cdot}248 \pm 0{\cdot}028$
Aswân „	$r = +0{\cdot}260 \pm 0{\cdot}039$
Gîza „	$r = +0{\cdot}277 \pm 0{\cdot}034$
Qina „	$r = +0{\cdot}286 \pm 0{\cdot}022$
Qalyubîya „	$r = +0{\cdot}303 \pm 0{\cdot}026$

* *Biometrika*, Vol. VIII. Nos. 1 and 2, July 1911.

Asyût Province . . .	$r = +0{\cdot}308 \pm 0{\cdot}021$
Gharbîya „ . . .	$r = +0{\cdot}316 \pm 0{\cdot}018$
Canal Governorate . . .	$r = +0{\cdot}330 \pm 0{\cdot}054$
Faîyûm Province . . .	$r = +0{\cdot}369 \pm 0{\cdot}029$

All the above coefficients of correlation are positive in sign and each exceeds three times its corresponding probable error. It will be seen from the above range that the differences are suggestive, therefore I am undertaking further investigations and hope to give the results shortly.

Table II, (*a*) to (*q*) gives the correlation-tables from which the constants have been calculated.

TABLE II. (*a*) *Alexandria.*

Length of Head.

Breadth of Head.	170—172	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	Totals
130—132	1	—	—	—	—	—	—	1	—	—	—	—	—	2
133—135	1	—	1	2	2	1	2	1	—	—	—	—	—	10
136—138	—	—	2	7	7	17	9	9	1	1	3	—	1	57
139—141	—	1	4	5	15	25	24	20	10	3	4	—	—	111
142—144	—	1	2	6	15	29	38	24	17	6	3	3	1	145
145—147	—	1	3	3	8	18	28	34	20	12	7	1	2	137
148—150	—	—	4	1	7	10	22	33	13	11	4	1	1	107
151—153	—	—	1	2	2	8	11	8	5	3	1	4	—	45
154—156	—	—	—	1	2	2	4	5	2	—	1	2	—	19
157—159	—	—	—	—	—	2	1	2	—	1	1	—	1	8
160—162	—	—	—	—	1	—	1	—	—	—	—	—	—	2
Totals	2	3	17	27	59	112	140	137	68	37	24	11	6	643

$r = +0{\cdot}244 \pm 0{\cdot}025.$

(*b*) *Cairo.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	Totals
130—132	—	1	—	1	1	1	1	—	1	—	—	—	6
133—135	—	—	—	2	2	2	2	—	—	—	—	—	8
136—138	3	3	10	9	9	10	15	8	8	—	—	—	75
139—141	—	1	5	14	17	29	35	8	4	4	3	—	120
142—144	—	2	9	16	24	54	31	24	19	8	1	2	190
145—147	1	1	9	17	39	40	41	29	12	11	1	1	202
148—150	—	—	3	7	19	32	22	13	18	8	1	2	125
151—153	—	—	5	1	2	6	11	13	7	5	1	4	55
154—156	—	—	—	2	1	1	2	2	2	1	—	1	12
157—159	—	—	—	—	—	2	—	1	4	1	—	—	8
160—162	—	—	—	—	—	—	—	—	—	1	—	—	1
Totals	4	8	41	69	114	177	160	98	75	39	7	10	802

$r = +0{\cdot}244 \pm 0{\cdot}025.$

(*c*) *Canal.*

Length of Head.

Breadth of Head.	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	Totals
133—135	2	—	1	—	2	—	—	—	—	—	5
136—138	—	3	1	1	3	—	2	1	—	—	11
139—141	—	1	3	4	4	5	1	1	—	—	19
142—144	—	—	2	4	4	7	1	—	2	1	21
145—147	—	—	5	4	6	8	4	4	2	—	33
148—150	—	1	3	—	4	2	5	2	—	—	17
151—153	—	—	—	1	—	2	2	3	2	1	11
154—156	—	—	—	1	—	1	1	3	—	—	6
157—159	—	—	—	—	1	1	—	—	—	—	2
160—162	—	—	—	1	—	1	—	—	—	—	2
Totals	2	5	15	16	24	27	16	14	6	2	127

$r = +0{\cdot}330 \pm 0{\cdot}054.$

(*d*) *Beheira.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	Totals
121—123	—	—	—	—	—	—	—	—	—	—	1	—	1
124—126	—	—	—	—	—	—	—	—	—	—	—	—	0
127—129	1	—	—	—	—	—	—	—	—	—	—	—	1
130—132	—	—	1	—	2	1	2	—	—	—	—	—	6
133—135	—	—	1	1	4	2	—	1	—	1	—	—	10
136—138	—	1	2	7	10	12	12	7	2	1	—	1	55
139—141	1	2	6	7	13	18	17	8	13	—	—	—	85
142—144	2	2	4	6	12	27	23	38	13	4	4	1	136
145—147	—	—	5	9	14	26	34	20	15	7	2	—	132
148—150	—	—	—	6	6	7	20	13	4	3	3	—	62
151—153	—	1	—	—	6	2	4	4	4	6	2	—	29
154—156	—	—	—	1	1	—	—	1	1	2	—	—	6
157—159	—	—	—	—	—	—	2	1	—	—	—	—	3
Totals	4	6	19	37	68	95	114	93	52	24	12	2	526

$r = +0{\cdot}213 \pm 0{\cdot}028.$

(*e*) *Gharbîya.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	215—217	218—220	Totals
127—129	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	2
130—132	1	—	—	1	1	2	2	—	—	1	—	—	—	—	—	—	8
133—135	1	—	6	5	7	9	5	3	2	—	—	—	—	—	—	—	38
136—138	1	2	9	13	23	27	20	9	10	—	—	—	—	—	—	—	114
139—141	1	2	14	18	42	48	40	29	18	6	1	1	—	—	—	—	220
142—144	—	2	9	25	26	62	45	44	16	6	1	1	—	—	—	—	237
145—147	1	6	2	14	41	53	51	48	31	22	6	1	1	—	—	—	277
148—150	—	1	3	4	20	20	27	19	21	12	6	—	—	—	—	1	134
151—153	—	—	—	3	2	7	12	14	9	2	2	—	—	—	—	—	51
154—156	—	—	—	—	—	3	2	4	2	6	—	—	—	—	—	—	17
157—159	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	2	4
160—162	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
163—165	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
166—168	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1
Totals	5	13	43	83	163	232	206	170	110	56	16	3	1	0	0	3	1104

$r = +0{\cdot}316 \pm 0{\cdot}018.$

(*f*) *Minufiya.*

Length of Head.

Breadth of Head.	170—172	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
130—132	—	—	—	1	1	1	—	1	—	—	—	—	—	—	—	4
133—135	—	—	1	2	4	4	6	2	1	1	4	1	—	—	—	26
136—138	1	1	1	1	6	17	24	13	1	1	2	—	1	—	—	69
139—141	—	1	—	3	22	23	26	25	21	9	5	2	1	—	—	138
142—144	—	—	1	9	8	27	26	35	31	13	10	3	1	1	—	165
145—147	—	1	—	6	10	26	38	39	27	17	13	2	—	—	1	180
148—150	—	—	—	6	4	11	13	17	19	10	7	3	1	1	—	92
151—153	—	—	—	1	—	4	2	4	8	6	5	—	—	—	—	30
154—156	—	—	—	—	—	1	1	—	2	3	2	1	—	—	—	10
157—159	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	2
160—162	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	1	3	3	29	55	116	136	136	110	60	49	12	4	2	1	717

$r = +0{\cdot}230 \pm 0{\cdot}024.$

(*g*) *Daqahliya.*

Length of Head.

Breadth of Head.	167—169	170—172	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	Totals
130—132	—	—	—	—	2	—	1	2	—	—	—	—	—	—	—	5
133—135	—	—	—	—	1	3	4	4	1	1	1	—	—	—	—	15
136—138	—	—	—	1	2	6	12	12	3	4	3	1	—	1	1	46
139—141	1	—	3	4	5	6	15	18	18	11	6	4	—	—	—	91
142—144	—	—	—	4	6	17	14	15	22	19	9	7	4	—	—	117
145—147	—	1	1	1	6	7	13	26	18	15	9	3	4	2	—	106
148—150	—	1	2	—	4	3	7	15	15	11	10	14	1	1	—	84
151—153	—	—	—	—	2	—	4	3	7	8	2	1	—	—	—	27
154—156	—	—	—	1	—	—	—	1	1	3	3	1	1	—	—	11
157—159	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	2
Totals	1	2	6	11	28	42	70	97	85	73	43	31	10	4	1	504

$r = +0{\cdot}212 \pm 0{\cdot}029.$

(*h*) *Sharqîya.*

Length of Head.

Breadth of Head.	164—166	167—169	170—172	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
127—129	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
130—132	—	—	—	1	—	—	—	1	1	—	—	—	—	—	—	—	—	3
133—135	—	—	—	—	1	1	4	2	4	2	4	2	1	—	—	—	—	21
136—138	—	—	—	—	1	3	6	5	11	7	5	8	—	1	1	—	—	48
139—141	—	—	—	—	1	10	3	11	32	19	9	5	1	3	—	1	—	95
142—144	—	—	—	—	1	4	12	26	24	22	17	9	4	2	—	—	—	121
145—147	1	—	—	—	2	2	11	20	17	27	18	10	3	2	1	1	—	115
148—150	—	—	—	—	2	1	3	5	18	17	23	9	4	3	—	—	—	85
151—153	—	—	—	—	—	—	1	2	3	1	4	1	2	—	—	—	1	15
154—156	—	—	—	—	—	—	—	—	—	2	2	—	2	—	—	—	—	6
157—159	—	—	—	—	—	—	—	—	—	—	1	2	—	1	—	—	—	4
160—162	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
163—165	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
166—168	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
Totals	2	0	0	1	8	21	40	72	110	97	83	46	17	12	3	2	1	515

$r = +0{\cdot}248 \pm 0{\cdot}028.$

(*i*) *Qalyubîya.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	Totals
130—132	—	—	—	1	—	—	1	—	—	—	—	—	2
133—135	—	1	1	2	2	2	1	2	—	—	—	—	11
136—138	—	—	1	4	7	7	5	2	2	—	—	—	28
139—141	—	1	1	6	8	12	7	7	4	2	—	—	48
142—144	1	1	3	7	11	18	14	9	9	5	2	—	80
145—147	—	1	2	5	10	17	15	13	4	3	3	1	74
148—150	—	—	—	5	3	4	7	9	2	3	—	—	33
151—153	—	—	—	—	—	1	3	2	—	—	1	—	7
154—156	—	—	—	—	—	1	2	2	1	2	—	—	8
157—159	—	—	—	—	—	—	1	—	—	1	1	1	4
Totals	1	4	8	30	41	62	56	46	22	16	7	2	295

$r = +0{\cdot}303 \pm 0{\cdot}026.$

(*j*) *Gîza.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
127—129	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
130—132	—	—	—	2	—	1	—	—	—	—	1	—	—	—	4
133—135	1	1	—	1	—	3	1	3	—	—	—	—	—	—	10
136—138	—	1	2	2	10	8	4	3	3	—	1	—	—	—	34
139—141	—	—	—	7	10	14	15	8	5	2	—	—	—	—	61
142—144	1	—	3	5	5	18	18	21	12	5	3	1	—	—	92
145—147	—	—	—	3	7	14	21	13	4	10	—	1	—	—	73
148—150	—	—	1	2	3	8	3	7	4	4	1	—	—	1	34
151—153	—	—	—	1	—	1	6	2	1	1	1	—	—	—	13
154—156	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1
157—159	—	—	—	—	—	—	—	1	1	1	—	—	—	—	3
Totals	2	2	6	24	35	67	69	58	30	23	7	2	0	1	326

$r = +0{\cdot}277 \pm 0{\cdot}034.$

(*k*) *Fatyûm.*

Length of Head.

Breadth of Head.	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
130—132	—	—	2	2	4	—	—	—	—	—	—	—	—	8
133—135	2	3	3	9	6	7	2	3	—	—	—	—	—	35
136—138	1	5	6	6	15	12	8	3	—	—	—	—	—	56
139—141	1	4	10	11	19	21	15	7	3	1	—	—	—	92
142—144	—	1	8	14	26	16	21	13	6	1	3	—	—	109
145—147	1	1	2	8	12	16	18	10	3	—	—	—	—	71
148—150	—	—	1	4	1	2	7	3	3	1	3	—	—	25
151—153	—	—	1	1	—	3	4	1	1	—	—	—	—	11
154—156	—	—	—	—	1	1	—	—	—	1	—	—	—	3
157—159	—	—	—	—	—	—	—	—	—	—	—	—	2	2
160—162	—	—	—	—	—	—	1	—	—	—	—	—	—	1
Totals	5	14	33	55	84	78	76	40	16	4	6	0	2	413

$r = +0{\cdot}369 \pm 0{\cdot}029.$

(*l*) *Beni Sueif.*

Length of Head.

Breadth of Head.	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	Totals
124—126	—	—	—	—	—	—	—	—	—	—	—	1	1
127—129	—	—	—	1	—	—	—	—	—	—	—	—	1
130—132	—	—	—	1	2	1	—	—	—	—	—	—	4
133—135	1	—	2	5	4	4	3	1	—	—	—	—	20
136—138	2	5	7	4	12	7	5	2	3	2	—	1	50
139—141	1	2	5	9	14	17	15	10	2	1	1	—	77
142—144	—	1	7	12	23	24	17	12	6	—	—	—	102
145—147	—	—	3	8	12	15	10	6	5	2	—	—	61
148—150	—	—	3	3	6	12	11	5	7	2	—	—	49
151—153	—	—	—	1	2	4	—	4	2	—	—	—	13
154—156	—	—	1	—	—	2	1	1	—	—	—	—	5
157—159	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	4	8	28	44	75	86	62	42	25	7	1	2	384

$r = +0{\cdot}208 \pm 0{\cdot}033.$

(*m*) *Minya.*

Length of Head.

Breadth of Head.	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	215—217	Totals
127—129	—	—	—	1	—	1	—	—	—	—	—	—	—	—	2
130—132	—	—	—	1	1	—	—	2	—	—	—	—	—	—	4
133—135	1	—	1	6	5	4	1	1	1	—	—	—	—	—	20
136—138	2	3	8	12	8	15	10	2	5	1	—	—	—	—	66
139—141	—	3	4	19	21	28	13	10	3	1	—	—	—	—	102
142—144	1	5	8	13	25	33	25	13	11	2	1	—	—	—	137
145—147	—	3	6	6	18	23	18	9	8	2	1	—	—	—	94
148—150	—	—	2	6	6	9	9	2	4	3	—	—	—	1	42
151—153	—	—	—	—	2	6	6	1	3	—	1	—	—	—	19
154—156	—	—	—	—	—	2	1	—	—	1	—	—	—	—	4
157—159	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
Totals	4	14	29	64	86	122	83	40	35	10	3	0	0	1	491

$r = +0{\cdot}227 \pm 0{\cdot}029.$

(*n*) *Asyût.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
130—132	1	1	1	2	1	2	—	1	—	—	—	—	—	—	9
133—135	—	—	1	6	14	7	9	5	3	—	1	—	—	—	46
136—138	—	7	9	19	21	23	16	11	3	5	—	—	—	1	115
139—141	—	4	5	14	33	46	45	16	11	7	1	1	—	—	183
142—144	1	2	2	15	38	48	59	33	24	14	3	1	—	—	240
145—147	—	1	4	9	22	33	35	36	20	11	5	2	—	1	179
148—150	—	—	2	3	10	15	22	12	9	4	4	2	—	—	83
151—153	—	—	—	3	—	—	2	7	10	—	3	—	—	—	25
154—156	—	—	—	—	1	—	—	1	3	1	—	—	—	—	6
157—159	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
Totals	2	15	24	71	140	174	188	122	84	42	17	6	0	2	887

$r = +0{\cdot}308 \pm 0{\cdot}021.$

(*o*) *Girga.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
127—129	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
130—132	—	1	1	1	3	1	1	—	—	—	—	—	—	—	8
133—135	—	—	2	2	5	9	10	2	1	—	—	—	—	—	31
136—138	—	—	4	9	14	19	22	7	7	4	1	—	—	1	88
139—141	1	1	5	11	18	23	22	13	17	8	—	—	1	1	121
142—144	—	—	5	16	22	33	34	29	10	7	4	1	—	—	161
145—147	—	1	1	9	16	21	28	21	17	12	1	5	1	—	133
148—150	—	—	1	2	2	9	9	13	4	7	3	1	1	—	52
151—153	—	—	—	—	—	5	3	1	2	—	2	—	—	—	13
154—156	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
157—159	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	3	20	50	81	120	129	86	59	38	11	7	3	2	610

$r = +0{\cdot}228 \pm 0{\cdot}026.$

(*p*) *Qina.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	Totals
130—132	—	1	1	2	3	5	—	1	—	—	—	—	—	13
133—135	1	4	4	9	10	5	7	5	—	—	1	—	—	46
136—138	1	5	7	10	14	33	23	14	12	3	2	—	—	124
139—141	1	2	7	13	26	40	31	25	14	11	5	1	—	176
142—144	—	1	4	20	24	46	41	33	29	6	8	1	—	213
145—147	—	1	2	4	13	21	34	27	17	8	—	2	—	129
148—150	—	1	2	4	6	13	23	19	11	12	2	1	1	95
151—153	—	—	—	—	2	4	3	3	2	—	—	1	—	15
154—156	—	—	1	—	1	—	1	2	1	2	—	1	—	9
157—159	—	—	—	—	—	—	—	2	1	—	—	—	—	3
160—162	—	—	—	—	—	—	—	1	—	—	—	—	—	1
Totals	3	15	28	62	99	167	163	132	87	42	18	7	1	824

$r = +0{\cdot}286 \pm 0{\cdot}022.$

(*q*) *Aswân.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	Totals
127—129	1	1	—	—	—	—	—	—	—	—	—	—	2
130—132	—	—	1	1	—	—	—	—	—	—	—	—	2
133—135	—	—	2	—	2	—	1	—	1	—	—	—	6
136—138	—	1	—	2	2	9	3	5	—	—	1	—	23
139—141	—	1	4	7	5	11	5	5	4	2	—	—	44
142—144	—	1	1	3	16	21	9	9	5	5	1	—	71
145—147	—	—	—	3	2	14	4	8	3	—	2	—	36
148—150	—	—	3	4	8	6	8	5	6	5	—	—	45
151—153	—	—	—	1	4	5	5	2	1	3	1	1	23
154—156	—	—	—	1	2	1	2	—	1	1	1	—	9
157—159	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	1	4	11	22	41	67	37	35	21	16	6	1	262

$r = +0{\cdot}260 \pm 0{\cdot}039.$

NOTES ON THE HISTORY OF CORRELATION.

Being a paper read to the Society of Biometricians and Mathematical Statisticians, June 14, 1920.

By KARL PEARSON, F.R.S.

(1) As I have often stated, Laplace anticipated Gauss by some 40 years. In his memoir of 1783, *Histoire de l'Académie*, pp. 423—467, he gives the expression for the probability integral

$$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{1}{2}x^2}dx$$

and suggests (p. 433) its tabulation as a useful task. It is clear that to do this is to recognise the existence of the probability-curve

$$y=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2},$$

or in its doubly projected form

$$y=\frac{N}{\sqrt{2\pi\sigma}}\,e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}.$$

Laplace's investigation while not proceeding from the very simple axioms of Gauss, which lead directly to the above equation, is more satisfactory than Gauss' because we see better the nature of the approximations by which the curve is reached and get hints of how to generalise it. Many years ago I called the Laplace-Gaussian curve the *normal* curve, which name, while it avoids an international question of priority, has the disadvantage of leading people to believe that all other distributions of frequency are in one sense or another 'abnormal.' That belief is, of course, not justifiable. It has led many writers to try and force all frequency by aid of one or another process of distortion into a 'normal' curve.

Gauss starting with a normal curve as the law of distribution of errors reached at once the method of least squares. To understand the origin of the correlational calculus we must really go back to Gauss' fundamental memoirs on least squares, namely the *Theoria combinationis observationum erroribus minimis obnoxiae* of 1823 and the *Supplementum* of 1826.

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We observe or measure *directly* a certain number of quantities $a, b, c, d, \ldots$. Each of these quantities is supposed by Gauss to be independent and to follow the normal law. The combined probability of the system* is accordingly

$$P \propto e^{-\frac{1}{2}\left\{\left(\frac{a-\bar{a}}{\sigma_a}\right)^2+\left(\frac{b-\bar{b}}{\sigma_b}\right)^2+\left(\frac{c-\bar{c}}{\sigma_c}\right)^2+\ldots\right\}},$$

or the product of the independent probabilities, where $\sigma_a, \sigma_b, \sigma_c, \ldots$ are the variability in errors of $a, b, c, \ldots$ and $\bar{a}, \bar{b}, \bar{c}, \ldots$ the means. This probability will be a maximum when

$$u^2_{a,\,b,\,c,\,\ldots} = S\left(\frac{a-\bar{a}}{\sigma_a}\right)^2$$

is a minimum. This is really the principle of weighted least squares. Its validity depends upon the normal law of distribution of error. Without this law holding it may be a utile method, but we have no means of proving it the 'best.'

The investigator in Gauss' case is, however, not interested in the quantities observed, but in certain indirectly ascertained quantities $x_1, x_2, \ldots x_n$ which are functions of them. Thus

$$\begin{aligned} x_1 &= f_1(a, b, c, \ldots), \\ x_2 &= f_2(a, b, c, \ldots), \\ \ldots &= \ldots\ldots\ldots\ldots, \end{aligned}$$

where $f_1, f_2, \ldots$ are known functions. Now Gauss cannot as a rule express from these general equations $a, b, c, \ldots$ in terms of $x_1, x_2, \ldots x_n$.

He assumes that all of them differ slightly from their mean or 'true' values and accordingly expands by Taylor's theorem and reaches the result†

$$\begin{aligned} x_1 - \bar{x}_1 &= \alpha_1(a-\bar{a}) + \beta_1(b-\bar{b}) + \gamma_1(c-\bar{c}) + \ldots, \\ x_2 - \bar{x}_2 &= \alpha_2(a-\bar{a}) + \beta_2(b-\bar{b}) + \gamma_2(c-\bar{c}) + \ldots, \end{aligned}$$

where the $\alpha, \beta, \gamma, \ldots$ are $\frac{df}{da}, \frac{df}{db}, \frac{df}{dc}, \ldots$ and can be ascertained *a priori*. Clearly Gauss supposes that a linear relationship is adequate, in other words he replaces statistical differentials by mathematical differentials, a step he does not really justify.

From these linear equations we can find the $a-\bar{a}, b-\bar{b}, c-\bar{c}, \ldots$ in terms of the indirectly observed variables $x_1-\bar{x}_1, x_2-\bar{x}_2, x_3-\bar{x}_3, \ldots$ by solution in determinantal form, say

$$\begin{aligned} a-\bar{a} &= A_1(x_1-\bar{x}_1) + B_1(x_2-\bar{x}_2) + C_1(x_3-\bar{x}_3)\ldots, \\ b-\bar{b} &= A_2(x_1-\bar{x}_1) + B_2(x_2-\bar{x}_2) + C_2(x_3-\bar{x}_3)\ldots. \end{aligned}$$

Substituting in u^2 we find

$$u^2_{x_1,\,x_2,\,\ldots} = S\left(\frac{A_1^2}{\sigma_a^2}\right)(x_1-\bar{x}_1)^2 + S\left(\frac{B_1^2}{\sigma_a^2}\right)(x_2-\bar{x}_2)^2 + 2S\left(\frac{A_1B_1}{\sigma_a^2}\right)(x_1-\bar{x}_1)(x_2-\bar{x}_2) + \ldots.$$

Hence the probability of $x_1, x_2, \ldots$ occurring is

$$P \propto e^{-\frac{1}{2}u^2_{x_1,\,x_2,\,\ldots}}.$$

* I use throughout notation which I assume now-a-days to be more familiar than that of Gauss.

† $\bar{a}, \bar{b}, \bar{c}, \ldots$ are actually in Gauss' method approximate or *guessed* solutions not means, but this does not affect the general nature of the discussion.

This is a normal surface which contains *the product terms*. As we now interpret it we say that the x's are *correlated* variates. And in this sense Gauss in 1823 reached the normal surface of n correlated variates. But he does not seek to express all his relations in terms of the S.D.'s σ_{x_1}, σ_{x_2}, σ_{x_3}, ... and the correlations r_{12}, r_{23}, ... of these variates. These x-variates are not for Gauss, nor for those who immediately followed him, the *directly* observed quantities. What he is seeking is the expression for σ_x, or the probable error of an indirectly observed variate in terms of

$$S\left(\frac{A_1^2}{\sigma_a^2}\right), \quad S\left(\frac{B_1^2}{\sigma_a^2}\right), \quad S\left(\frac{A_1 B_1}{\sigma_a^2}\right), \ldots.$$

In this case A, B, C are ratios of minors and determinants of the α, β, γ, ... which are Gauss' known quantities. His object therefore is to express σ_x not from direct observations but in terms of α, β, γ, ... through the sums of determinantal terms.

Writers on Least Squares and Adjustment of Observations then take w any function of x_1, x_2, ... x_n, i.e.

$$w = F(x_1, x_2, \ldots x_n),$$

express the relation in a linear form, i.e.

$$w - \overline{w} = \lambda_1 (x_1 - \bar{x}_1) + \lambda_2 (x_2 - \bar{x}_2) + \ldots,$$

and then, to find σ_w^2, go through lengthy analysis to determine

$$\text{Mean } (x_1 - \bar{x}_1)^2, \quad \text{Mean } (x_2 - \bar{x}_2)^2, \quad \text{Mean } (x_1 - \bar{x}_1)(x_2 - \bar{x}_2), \text{ etc.}$$

in terms of the original α, β, γ, ... There is not a word in their innumerable treatises that what is really being sought are the mutual correlations of a system of correlated variables. The mere using of the notation of the correlational calculus throws a flood of light into the mazes of the theory of errors of observation. There is much more in the theory of least squares than I have stated; there are equations of conditions—the angle and side equations of geodesy, etc.—these only complicate the matter. The point is this: that the Gaussian treatment leads (i) to a non-correlated surface for the directly observed variates, (ii) to a correlation surface for the *indirectly* observed variates. This occurrence of product terms arises from the geometrical relations between the two classes of variates, and not from an organic relation between the indirectly observed variates appearing on our direct measurement of them.

It will be seen that Gauss' treatment is almost the inverse of our modern conceptions of correlation. For him the *observed* variables are independent, for us the observed variables are associated or correlated. For him the non-observed variables are correlated owing to their known geometrical relations with observed variables; for us the unobservable variables may be supposed to be uncorrelated causes, and to be connected by unknown functional relations with the correlated variables. In short there is no trace in Gauss' work of observed physical variables being—apart from equations of condition—associated organically which is the fundamental conception of correlation.

(2) The next important work to be considered is that of August Bravais. It is entitled "Sur les probabilités des erreurs de situation d'un point." It was published in the *Mémoires présentés par divers savants à l'Académie royale des Sciences de l'Institut de France*, T. IX. Paris, 1846, pp. 256—332. It appears, however, to have been reported favourably upon in 1838*. Bravais was in many respects a remarkable man. Essentially a geologist he wrote also on astronomy, physics, meteorology and the theory of probabilities. He made a voyage to Lapland for geodesic purposes and took the opportunity of measuring a number of Lapp skulls! He had a width of action most sympathetic to the biometrician.

Writing in 1895 of the history of correlation I said:

"The fundamental theorems of correlation were for the first time and almost exhaustively discussed by Bravais [Title as above of his memoir] nearly half a century ago. He deals completely with the correlation of two and three variables." Then speaking of Galton's coefficient of correlation I say: "This indeed appears in Bravais' work, but a single symbol is not used for it. It will be found of great value in the present discussion. In 1892 Professor Edgeworth, also unconscious of Bravais' memoir, dealt in a paper on 'Correlated Averages' with correlation for three variables (*Phil. Mag.* Vol. XXXIV. 1892, pp. 194—204). He obtained results identical with Bravais', although expressed in terms of 'Galton's functions'" [i.e. coefficients of correlation].

Again later, p. 287, in giving the fundamental equation for the correlation of three variates I wrote: "This agrees with Bravais' result, except that he writes for r_1, r_2, r_3 the values $\Sigma(yz)/(n\sigma_2\sigma_3)$ etc., which we have shown to be the best values (see *loc. cit.* p. 267)." Again on p. 301 I write before proving the general theorem of multiple correlation: "*Edgeworth's Theorem.* We may stay for a moment over the results above to deduce Professor Edgeworth's Theorem," with the footnote, "Briefly stated with some rather disturbing printer's errors in the 'Phil. Mag.' Vol. XXXIV. p. 201, 1892."

Now all these statements if they were correct would indicate that Bravais discovered correlation before Galton and that Edgeworth first published the form of the multiple correlation surface. They have been accepted by later writers, notably Mr Yule in his manual of statistics, who writes (p. 188):

"Bravais introduced the product-sum, but not a single symbol for a coefficient of correlation. Sir Francis Galton developed the practical method, determining his coefficient (Galton's function as it was termed at first) graphically. Edgeworth developed the theoretical side further and Pearson introduced the product-sum formula."

Now I regret to say that nearly the whole of the above statements are hopelessly incorrect. Bravais has no claim, whatever, to supplant Francis Galton as the discoverer of the correlational calculus. For the most part he is simply taking a very special case of the Gaussian analysis, and nowhere on p. 267 of his memoir can I now find that he has used the expressions for the correlation symbols without

* *Comptes rendus*, T. VII. p. 77.

their names. Again Edgeworth did *not* obtain results identical with Bravais', he went on a route of his own to find the true multiple correlation surface and gave as I said in 1895 only doubtful results. But I fear they were not all due to printer's errors. On re-examining his memoir 25 years later I think he harnessed imperfect mathematical analysis to a jolting car and drove it into an Irish bog on his road, and that it was doubtful analysis not errors of printing which led to his obscure conclusions. I was scarcely justified in 1895 in calling the multiple regression result Edgeworth's Theorem. He had tried in 1892 to solve the problem, and he can hardly be said to have succeeded properly. It is very difficult to explain now how my errors of ascription came about, still less possible is it to understand why later writers have not corrected my false history, but merely repeated it.

As far as I can remember what happened at all, it was as follows. I know that I was immensely excited by Galton's book of 1889—*Natural Inheritance*—and that I read a paper on it in the year of its appearance. In 1891—2 I lectured popularly on probability at Gresham College, taking skew whist contours as illustrations of correlation. In 1892 I lectured on variation, in 1893 on correlation to research students at University College, the material being afterwards published as the first four of my *Phil. Trans.* memoirs on evolution. At this time I dealt with correlation and worked out the general theory for three*, four and ultimately n variables. The field was very wide and I was far too excited to stop to investigate properly what other people had done. I wanted to reach new results and apply them. Accordingly I did not examine carefully either Bravais or Edgeworth, and when I came to put my lecture notes on correlation into written form, probably asked somebody who attended the lectures to examine the papers and say what was in them. Only when I now come back to the papers of Bravais and Edgeworth do I realise not only that I did grave injustice to others, but made most misleading statements which have been spread broadcast by the text-book writers.

(3) Let us now examine Bravais' memoir. He commences by stating that he is going to measure the errors of the determination of the coordinates x, y, z of a point in space. These coordinates are not measured directly but are functions of the observed elements $a, b, c, \ldots$, and he puts

$$x = \phi(a, b, c, \ldots),$$
$$y = \psi(a, b, c, \ldots),$$
$$z = \chi(a, b, c, \ldots).$$

He then expands x, y, z linearly in terms of a, b, c assuming that mathematical differentials may be used for errors; thus he writes

$$\delta x = A\delta a + B\delta b + C\delta c + \ldots,$$
$$\delta y = A'\delta a + B'\delta b + C'\delta c + \ldots,$$
$$\delta z = A''\delta a + B''\delta b + C''\delta c + \ldots.$$

He tells us that the A, B, C are differential coefficients, i.e. of the known functions ϕ, ψ, χ, and that to justify the neglect of higher powers and products we must get

* Published in the *R. S. Proc.* Vol. LVIII. p. 241, 1895.

rid of constant sources of error which arise chiefly from vices of method of observation, ignorance of physical laws, etc. That they can be removed by increasing the number of our observations, and in surveying—which he has essentially in mind—by using the repeating circle, which destroys the majority of constant errors and lessens the influence of variable causes by the fact itself of repeating the observed angles. It is clear that he is thinking solely of theodolite work, and that his x, y, z are Gauss' indirectly observed quantities, his directly observed quantities being angles and bases a, b, c, ...

He now changes his notation; he uses x, y, z for the errors δx, δy, δz, and m, n, p for δa, δb, δc, and takes equations

$$x = Am + Bn + Cp + \ldots$$

and calls x, y, z the *dependent* variables, m, n, p the *independent* variables. He says that Laplace has shown that a variation of x between x and $x + \delta x$ will be of the form

$$\sqrt{\frac{h_x}{\pi}}\, e^{-h_x x^2}\, dx,$$

where h_x is given by

$$\frac{1}{h_x} = \frac{A^2}{h_m} + \frac{B^2}{h_n} + \frac{C^2}{h_p} + \ldots.$$

It is therefore clear that he supposes that his *observed* quantities m, n, p, ... are *uncorrelated* in our sense of the word. In fact he gives for two and three variates the expressions

$$\sqrt{\frac{h_m h_n}{\pi . \pi}}\, e^{-(h_m m^2 + h_n n^2)}\, dm\, dn \quad \text{(p. 261)},$$

$$\sqrt{\frac{h_m h_n h_p}{\pi . \pi . \pi}}\, e^{-(h_m m^2 + h_n n^2 + h_p p^2)}\, dm\, dn\, dp \quad \text{(p. 264)}.$$

There is obviously not a single step, not a line in this, which does not occur in Gauss, except that Gauss would use

$$u^2 = h_m m^2 + h_n n^2 + h_p p^2$$

and not trouble to state that the probability was given by the exponential.

Now Gauss' problem was to express the variability of x in terms of the variability of the observed quantities a, b, c, ... or m, n, p, and of the differential coefficients A, B, C. This is absolutely the same as Bravais' problem, and Bravais' treatment goes very little further than Gauss'—indeed it is essentially narrower as while Gauss neither limits the number of his variables nor their nature, Bravais treats only of *position* in space.

I will now give the value of the expression Bravais reaches for his surface of two dimensions, expressing by d^2w the briquette of frequency on $dx dy$:

$$\frac{d^2w}{dx\,dy} = \frac{1}{\pi} \frac{1}{\left\{\Sigma \dfrac{(AB' - A'B)^2}{h_m h_n}\right\}^{\frac{1}{2}}} e^{-\dfrac{\left\{x^2 \Sigma \dfrac{A'^2}{h_m} - 2xy \Sigma \dfrac{AA'}{h_m} + y^2 \Sigma \dfrac{A^2}{h_m}\right\}}{\Sigma \left(\dfrac{(AB' - A'B)^2}{h_m h_n}\right)}} \quad \text{(p. 272)}.$$

Now, if you take

$$x = Am + Bn + \ldots,$$
$$y = A'm + B'n + \ldots,$$
$$\sigma_x^2 = A^2\sigma_m^2 + B^2\sigma_n^2 + \ldots,$$
$$\sigma_y^2 = A'^2\sigma_m^2 + B'^2\sigma_n^2 + \ldots,$$
$$\text{Mean}\,(\delta x\,\delta y) = r_{xy}\sigma_x\sigma_y = AA'\sigma_m^2 + BB'\sigma_n^2 + \ldots.$$

Whence

$$\sigma_x^2\sigma_y^2(1 - r_{xy}^2) = \Sigma\,(A^2B'^2 + A'^2B^2 - 2AA'BB')\,\sigma_n^2\sigma_m^2$$
$$= \Sigma\,(AB' - A'B)^2\,\sigma_n^2\sigma_m^2,$$

whence, remembering $\quad h_m = \dfrac{1}{2\sigma_m^2}, \quad h_n = \dfrac{1}{2\sigma_n^2},$

we easily deduce the

$$z = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - r_{xy}^2}}\, e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} - \frac{2r_{xy}xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\frac{1}{1-r_{xy}^2}}$$

of our familiar notation.

But this is precisely what Bravais does *not* do, and for the simple reason that his x, y, z are not variables which he has directly determined and for which he can directly find σ_x, σ_y and r_{xy}. He is merely seeking to express the variability of x and y in terms of the directly determined constants and certain differential coefficients. This is one of the fundamental problems of the Method of Least Squares and had already been solved by Gauss. Bravais adds so far *nothing whatever* to Gauss' solution of 20 years earlier. If Bravais discovered correlation, then Gauss had done so previously.

As a matter of fact while the above expression shows how a hasty examination of Bravais' memoir might lead one to believe he had reached the correlation surface, he was in fact occupied with an entirely different problem, one which was really only a particular case of Gauss' earlier and more comprehensive work.

We cannot pass over, however, the really valuable portion of Bravais' memoir. It lies in this: Having got his coefficients of x and y in terms of the differential coefficients A, B, C, ... he writes the surface

$$\frac{K}{\pi}\, e^{-K^2(\alpha_1 x^2 - 2\beta_0 xy + \alpha_2 y^2)},$$

and then discusses the properties of a surface of which the contours are

$$ax^2 + 2cxy + by^2 = D,$$

i.e. the familiar ellipses of our normal surface. He gets the conjugate of x-axes as the locus of maximum y's and determines the probability of points lying in certain areas—bounded by similar ellipses or in angular sectors. He gets the line $x = -\dfrac{c}{a}y$, which corresponds to Galton's regression-line. But this is not a result of observing x and y and determining their association, but of the fact that x and y

are functions of certain independent and directly observed quantities. When he thinks of c and a at all, it is not in terms of observations on x and y but of the differential coefficients A, B, C of the geometrical relations between position in space and the angles by which that position is found.

Next we come to his surface of three variates and the treatment is identical. He writes

$$z = \frac{G}{\pi^{\frac{3}{2}}} e^{-(ax^2+by^2+cz^2+2ezy+2fxz+2gxy)} \quad \text{(p. 296)},$$

and his primary object is to determine a, b, c, e, f, g in terms of the differential coefficients A, B, C and the variabilities of the observed independent variates.

Thus he gives

$$a = G^2\Sigma \frac{1}{h_m h_n} \{(A'B'')\}^2, \quad e = G^2\Sigma \frac{1}{h_m h_n} \{(A''B)(A''B')\}, \quad \frac{1}{G^2} = \Sigma \frac{\{(AB'C'')\}^2}{h_m h_n h_p}.$$

There is throughout merely the standpoint of the Gaussian method of treating errors of observation, and if we are to attribute any discovery of the idea of correlation to Bravais we must with the same confidence assert that Gauss was the primary originator of the whole idea. To my mind this is absurd*. In the case of both these distinguished men the quantities they were observing were *absolutely independent*; they neither of them had the least idea of correlation between observed quantities. The product terms in their expressions—never analysed in the sense of correlation—arise solely not from organic relationships, but from the *geometrical* relationships which exist between their observed quantities and the indirectly observed quantities they deduce from them. Bravais himself (p. 331) says that the application of his results are narrowly circumscribed by the nature of his assumptions—astronomy and the great geodesic surveys alone provide sufficiently accurate material. As far as Gauss and Bravais are concerned we must, I think, hold that they contributed nothing of real importance to the problem of correlation, and that my statement of 1895 was a totally erroneous one.

The same criticism applies to all the treatment of the normal surfaces by later writers, which are described at very considerable length by Czuber in his *Theorie der Beobachtungfehler*, Leipzig, 1891. In all cases the variables are *indirectly* observed quantities and the product terms arise because they are mathematically supposed to be linear functions of the directly observed, but quite independent variables. That the directly measured quantities might themselves be correlated does not seem to have occurred to the many writers on the theory of observations.

As far as I am aware there is nothing to record on our subject beyond the work of the writers on the theory of observations referred to above until we reach Francis Galton himself. His first statement of his ideas was in a lecture at the

* I feel quite certain that if any one had told either Gauss or Bravais that $\Sigma(ab)$ for their *observed* measurements need not be zero, they would have been laughed out of court, as the astronomers now laugh at us, when we assert that their measurements of different stellar magnitudes are very probably correlated!

Royal Institution, Feb. 9, 1877. He had found it very difficult to collect human material for two generations and after careful consideration selected sweet pea seeds. These seeds were both measured and weighed and actually observations were taken on foliage and length of pod although as far as I am aware Galton never published the reductions of the latter. As he himself writes in 1885: "It was anthropological evidence that I desired, caring only for the seeds as means of throwing light on heredity in man. I tried in vain for a long and weary time to obtain it in sufficient abundance, and my failure was a cogent motive, together with others, in inducing me to make an offer of prizes for family records, which was largely responded to, and furnished me last year with what I wanted*."

The title of Galton's R. I. lecture was *Typical Laws of Heredity in Man*. Here for the first time appears a numerical measure r of what is termed 'reversion' and which Galton later termed 'regression.' This r is the source of our symbol for the correlation coefficient, which was really the first letter of 'reversion' not of 'regression.' The main results are given in a mathematical appendix†. Galton works with the modulus—i.e. our $\sqrt{2}\sigma$—probably because the tables of the probability integral were then given in the modulus as argument. But we can at once convert into more customary notation. Thus we find the now familiar result

$$v = \sqrt{1 - r^2}\, c_1,$$

or, translating his symbols:

$$\text{Variability of family} = \sqrt{1 - r^2} \times \text{variability of general population.}$$

Galton had already reached the idea of homoscedasticity in the arrays of offspring. "I was certainly astonished to find the variability of the produce of the little seeds to be equal to that of the big ones; but so it was and I thankfully accept the fact, for had it been otherwise, I cannot imagine, from theoretical considerations, how the typical problem could be solved" (p. 10).

Next Galton supposes the mean taken of both parents and notes that the "variability of the parentage," what he would have called later the mid-parentage, $= \frac{1}{\sqrt{2}}$ variability of either parent. He has not yet reached the idea of reducing one sex to the standard of the other, and the result is only true, if we have to deal with characters not sexually differentiated.

Now we come to the test point‡:

"Reversion"—Galton tells us, p. 10—"is the tendency of the ideal mean filial type to depart from the parental type, reverting to what may be roughly and perhaps fairly described as the average ancestral type. If family variability had been the only process in simple descent that affected the characteristics of a sample the dispersion of the race from its mean ideal type would indefinitely increase with

* Address to Anthropological Section, *B. A. Report*, 1885, p. 1207.

† Royal Institution of Great Britain, Friday, February 9, 1877.

‡ Let the reader remember that these words were spoken just 40 years ago, and that they waited 12 to bring forth fruit!

the number of generations, but reversion checks this increase, and brings it to a standstill."

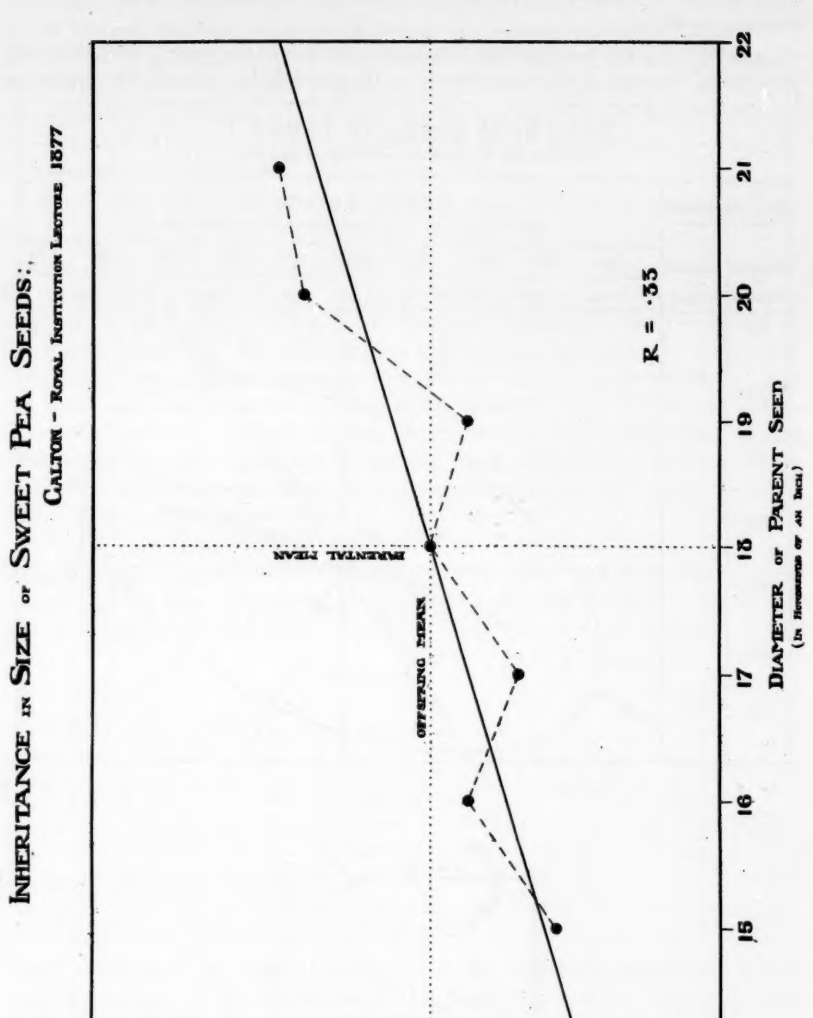
Galton's proof assuming homoscedasticity is of a very simple nature. Let the reversion be λx, where x is the parental character. Then the mean variability of the offspring generation

$$= \sigma^2 (1 - r^2) + \lambda^2 (\text{mean } x^2)$$
$$= \sigma^2 (1 - r^2) + \lambda^2 \sigma^2.$$

Therefore unless $\lambda = r$ the population cannot remain stable. Or without any hypothesis as to normality, only on the basis of linearity of reversion, homoscedasticity and stability the Galton coefficient r of reversion must be equal to the r which gives the reduction of the 'family variability.' Thus the lecture of 1877, while it contains points which later work was to clear up, still in the main lines gives on the data for size in sweet pea seeds the fundamental properties of the regression line. I have worked out Galton's data for sweet peas and show you a diagram of the result which Miss A. Davin has prepared for me. The parent seed was of course selected seed, and Galton took 100 of each parental grade and determined the mean of the offspring, which of course were non-selected seed, i.e. not seedsman's seed. Galton fixes the regression in round numbers at $\frac{1}{3}$, I make it slightly larger. In any case the regression coefficient is small, if we consider the sweet pea, as Galton did, as self-fertilising. It has been so proclaimed in several botanical investigations on heredity in the sweet pea. But in 1907 I watched a row of sweet peas and observed *Megachile Willughbiela*, the leaf-cutting bee, in quite considerable numbers visiting the flowers. The Superintendent of the R. H. S.'s garden at Wisley also replied to an inquiry that he had no doubt some English insect cross-fertilised sweet peas because in trying new sorts the gardeners had to place the rows in different parts of the garden to reduce the risk of cross-fertilisation. Darwin's statement* that "in this country it"—the sweet pea—"seems invariably to fertilise itself," appears open to question. Galton's coefficient may therefore, although low, be not so low as it appears on the assumption of self-fertilisation.

The next few years Galton was occupied in collecting material for further investigation of regression and heredity. He had established his Anthropometric Laboratory at South Kensington and by offering prizes obtained his Records of Family Faculties. The first-fruits of these data are to be found in his Presidential Address to the Anthropological Section of the British Association at Aberdeen in 1885. The part of this Address dealing with regression was considerably extended in a paper read to the Anthropological Institution in the same year. Galton now deals with the inheritance of stature and transmutes female to male stature before determining his mid-parentages. He does this, not as we should do now by multiplying by the ratio of paternal and maternal standard deviations, but by the multiplying factor of mean statures 1·08. This is roughly permissible if the coefficients of variation for the two sexes are the same as they very nearly are for stature. In this paper we have the first published diagram of the *two* regression

* *Cross and Self-fertilisation of Plants*, 1878, p. 153.


INHERITANCE IN SIZE OF SWEET PEA SEEDS.
GALTON - ROYAL INSTITUTION LECTURE 1877
R = ·33
PARENTAL MEAN
OFFSPRING MEAN
DIAMETER OF OFF SPRING SEED
(IN HUNDREDTHS OF AN INCH)
DIAMETER OF PARENT SEED
(IN HUNDREDTHS OF AN INCH)
19
18
17
16
15
14
22
21
20
19
18
17
16
15
14

lines and the first correlation table (of stature in parentage and offspring) as we should now call it.

Also Galton gives the diagram* which indicates how he discovered observationally the form of the normal frequency surface. He proceeded to smooth his correlation

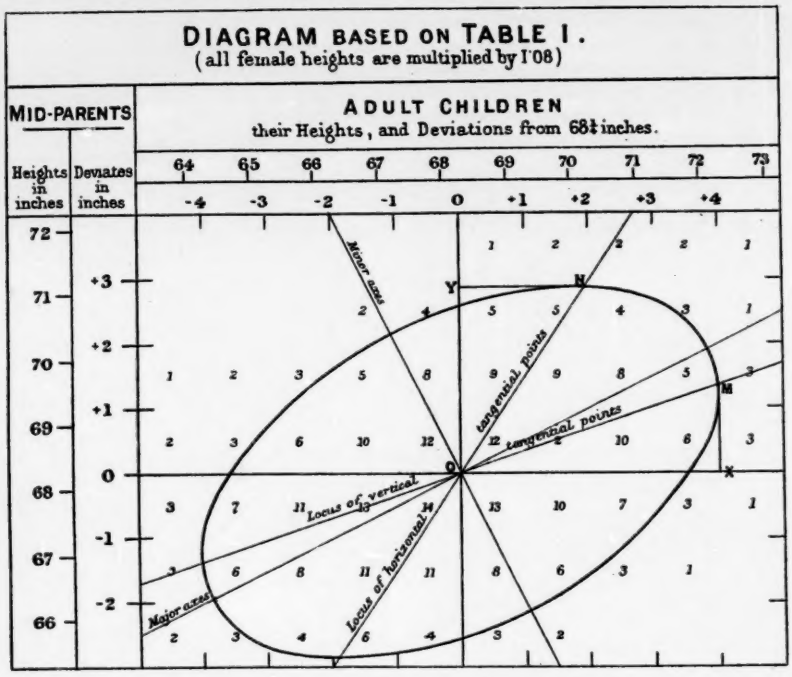


Fig (a)

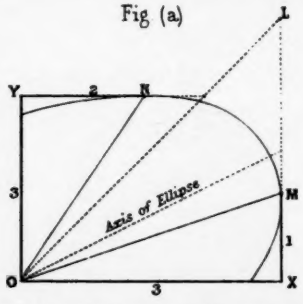


table by taking the mean of four adjacent cells, and then drawing contour lines through points of the same frequency. He found such contour lines were a system of concentric similar and similarly placed ellipsoids and that the regression lines

* Reproduced here by permission of the Royal Anthropological Institute.

were what the mathematician terms the conjugate diameters of the variate axes. He discovered that the sections parallel to the variate axes were 'apparently' normal curves of equal S.D. but that this S.D. was reduced and bore a constant ratio to the S.D. of the general population. He knew 8 years earlier the relation of $\sigma\sqrt{1-r^2}$ to the 'reversion' coefficient r. That Galton should have evolved all this from his observations is to my mind one of the most noteworthy scientific discoveries arising from pure analysis of observations.

Why Galton did not at once write down the equation to his surface as

$$z \propto e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}} \times e^{-\frac{1}{2}\frac{1}{\sigma_x^2(1-r^2)}\left(x - r\frac{\sigma_x}{\sigma_y}y\right)^2}$$

has always been a puzzle to me. Actually he carried the problem, stated in the language of probability, to Mr J. D. Hamilton Dickson, a mathematician of Peterhouse, Cambridge, who after stating the wording of Galton's problem, wrote down the answer substantially as above in the fourth line of his memoir*! The fact is that Galton's statement of his problem, involving as it did the assumption of normal distribution, homoscedasticity and linear regression, provided the answer the moment his results were read in symbols. The explanation of Galton's action possibly lies in the fact that Galton was very modest and throughout his life underrated his own mathematical powers.

Thus in 1885 Galton had completed the theory of bi-variate normal correlation. The next stage in the theory of correlation, multi-variate correlation, was directly indicated by the general problem of ancestry. As is now well known the fundamental regression equation is

$$\frac{x_0 - \bar{x}_0}{\sigma_0} = -\Sigma\left(\frac{R_{0s}}{R_{00}}\frac{x_s - \bar{x}_s}{\sigma_s}\right) \quad \text{.....(i),}$$

where R_{pq} is the p, q minor of the determinant

$$R = \begin{vmatrix} 1 & r_{01} & r_{02} & \dots & r_{0n} \\ r_{10} & 1 & r_{12} & \dots & r_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ r_{n0} & r_{n1} & r_{n2} & \dots & 1 \end{vmatrix}$$

and the variability of the array is

$$\sigma_0\sqrt{\frac{R}{R_{00}}} \quad \text{.....(ii).}$$

Galton endeavoured to reach this by a short cut, and thus evolved his law of ancestral heredity. This was a brilliant and suggestive step, but he was not able to state the conditions under which it is theoretically correct or bring forward data at that time to confirm its observational accuracy.

* *R. S. Proc.* Vol. XL. p. 63, 1886. Galton himself writes (*B. A. Report*, 1885, p. 1211), "I may be permitted to say that I never felt such a glow of loyalty and respect towards the sovereignty and magnificent sway of mathematical analysis as when his answer reached me, confirming, by purely mathematical reasoning, my various and laborious statistical conclusions with far more minuteness than I had dared to hope, for the original data ran somewhat roughly, and I had to smooth them with tender caution."

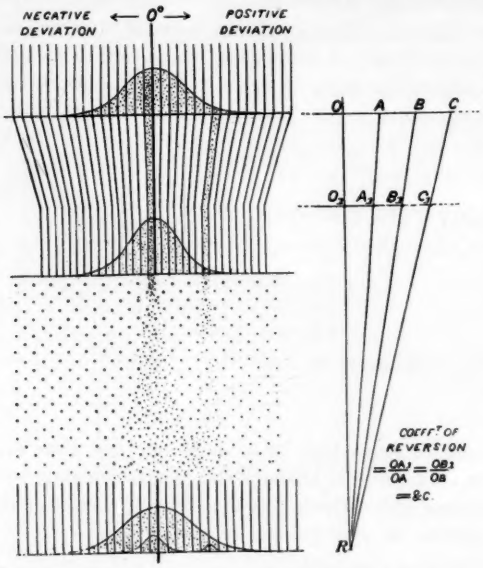
Another feature of Galton's work at this time must be noted. He worked with the median instead of the mean, and he used probable errors or quartile values instead of standard deviations. Further, to obtain r, he somewhat laboriously expressed both variates in terms of their quartile deviations: thus r became the slope of his regression line. It was then determined by graphically fitting a good line, or from certain chosen arrays. Thus he worked with somewhat primitive statistical tools, and the wonder is that he achieved as much by their aid as he did.

Given A and B with regression r_{ab}, B and C with regression r_{bc}, Galton assumed $r_{ac}=r_{ab}\times r_{bc}$ to obtain his kinship relations. A nephew is the son of a brother. Hence

$$r \text{ for uncle and nephew} = r \text{ for brothers} \times r \text{ for father and son.}$$

This of course is incorrect; it implies the vanishing of the corresponding partial correlation coefficient. Again, I think, his mid-parental correlation is not theoretically consonant with his parental correlations.

Another noteworthy point of the 1877 R. I. and the 1885 A. I. papers is the ample provision of mechanical models to illustrate by dropping shot or seeds the properties of bi-variate frequency. One wonders whether these elaborate quincunxes have been preserved and if so where they are at the present time. I reproduce one of them by permission from the *Journal of the Royal Institution*.



In 1886 Galton published a paper in the *Royal Society Proceedings** on "Family Likeness in Stature." This contains Hamilton Dickson's note and further data from Galton's *Family Records*.

* Vol. XL. pp. 42—66.

He gives under the headings "Mean regression w," and "Quartile of individual variability" the coefficients of correlation of various pairs of relations: Midparent and Offspring, Brothers, Fathers and Sons, Uncles and Nephews, Grandparents and Grandsons, but he does not realise that on the theory of multiple regression there are certain inconsistencies in his values. I do not think that there is much additional contribution to theory in this paper.

In 1888, however, Galton took a great step forward. He recognised that the whole statistical apparatus he had evolved for the treatment of the problem of heredity had a vastly wider significance. In a paper read to the Royal Society on December 5, 1888*, entitled "Correlations and their Measurement chiefly from Anthropometric Data," the term correlation first appears in our subject. Thus Galton's opening lines run:

"Co-relation or correlation of structure" is a phrase much used in biology, and not least in that branch of it which refers to heredity, and the idea is even more frequent than the phrase; but I am not aware of any previous attempt to define it clearly, to trace its mode of action, or to show how to measure its degree.

Two variable organs are said to be correlated when the variation of the one is accompanied on the average by more or less variation of the other, and in the same direction (p. 135).

The last words seem to us now out of place, but Galton had not yet reached the idea of negative correlation. Also the balance is still swinging between 'co-relation' and 'correlation' although it has ultimately fallen to the more weighty word. How clearly Galton grasped the essence of correlation may be shown by the following sentences which might have saved many ingenious later investigators thinking they had made an important discovery. "It is easy to see that co-relation must be the consequence of the variations of the two organs being partly due to common causes. If they were wholly due to common causes, the co-relation would be perfect, as is approximately the case with the symmetrically disposed parts of the body. If they were in no respect due to common causes, the co-relation would be *nil*. Between these two extremes are an endless number of intermediate cases, and it will be shown how the closeness of co-relation in any particular case admits of being expressed by a single number" (p. 135). This single number it is needless to say is our present coefficient of correlation. Galton drops now the w of his 1886 work and returns to the r of his 1877 lecture, and the symbol r has remained to the present day.

Galton's process is the same as in the heredity problem. He used median and quartile and reduces the deviations to their respective quartiles as unit. He then smooths his means of arrays, draws a line to represent them and reads off its slope as r. He thus determines seven correlations which he here terms "indices of correlation†." They are between Stature and Cubit, Stature and Head Length, Stature and Middle Finger Length, Cubit and Middle Finger Length, Head Length and Head Breadth, Stature and Height of Knee, Cubit and Height of Knee. He fully

* *R. S. Proc.* Vol. XLV. pp. 135—145.

† On p. 143 r, the index of co-relation, is identified with the 'regression' or 'reversion' of Galton's earlier papers.

realises (1) that r is the same when obtained from either variate as 'relative,' (2) that r is always less than unity, (3) that r measures the closeness of co-relation, and (4) provides the regression line (p. 145).

On p. 144 the term "partial co-relation" is used but hardly in our modern sense although Galton is feeling his way towards multiple correlation. One problem he gives on p. 144 perhaps deserves mention, namely, if the n variates be expressed in terms of their quartiles then the quartile variability of their sum is $\sqrt{n}$ if they are independent and n if they be "rigidly and perfectly co-related." "The actual value would be almost always somewhere intermediate between these extremes, and would give the information that is wanted."

What Galton needs is the "multiple correlation coefficient," i.e.

$$\sqrt{1-\frac{R}{R_{00}}},$$

but he is not yet on the right track for it.

In 1889 appeared Galton's book *Natural Inheritance* embodying most of the work we have discussed in the earlier memoirs of 1877 to 1888. Beyond this Galton did not carry the subject of correlation. He, in my opinion to-day, created it; there is nothing in the memoirs of Gauss or Bravais that really antedates his discoveries. They were dealing with the relatively narrow problem of determining the probable errors of indirectly observed quantities deduced from independent or uncorrelated directly observed quantities. The product-terms that arise in their investigations were expressed in terms of differential coefficients; they were not treated as a means of determining organic relationships between directly measured variates. Galton, starting from the organic relationship between parent and offspring, passed to the idea of a coefficient measuring the correlation of all pairs of organs, and thence to the 'organic' relationship of all sorts of factors. If you think Galton did not appreciate the width of his new methods you must turn to the last paragraph of his *Introduction* to the *Natural Inheritance*.

"The conclusions cannot, however, be intelligently presented in an introductory chapter. They depend on ideas that must first be well comprehended, and which are now novel to the large majority of readers and unfamiliar to all. But those who care to brace themselves to a sustained effort, need not feel much regret that the road to be travelled over is indirect, and does not admit of being mapped beforehand in a way they can clearly understand. It is full of interest of its own. It familiarizes us with the measurement of variability, and with curious laws of chance *that apply to a vast diversity of social subjects*. This part of the inquiry may be said to run along a road on a high level that affords wide views in unexpected directions, and from which easy descents may be made to totally different goals to those we have now to reach. I have a great subject to write upon..." (p. 2).

Galton realised as fully as any of us now the width of application that would open up to the new calculus of correlation, and what easy descents there would be from the "high level road" to strange goals. His notebooks of this period show

that he was applying correlation and the regression line in a variety of ways thus to the relation between wing and tail length in birds, to fertility and to disease. His advance was chiefly hampered by the restriction of his data and the need for organised observers and computers.

The publication of *Natural Inheritance* provided Francis Galton with at least three recruits for the field of correlation: Weldon, Edgeworth and myself.

Weldon started in 1889 measuring the organs of shrimps at Plymouth and he was able to announce early in 1890—the letter is now in the glass case in our library here—the first correlation coefficients, or as he termed them "Galton Functions," between organs in shrimps. This was rapidly followed by his work on crabs, and the attempt to show that Galton functions were the same for all local races of the same species. In his first paper on shrimps Weldon writes*:

"In making this investigation I have had the great privilege of being constantly advised and helped in every possible way by Mr Galton. My ignorance of statistical methods was so great that without Mr Galton's constant help, given by letter at the expenditure of a very great amount of time and trouble, this paper would never have been written."

The pupil, however, was soon to outdistance the master in the width of his theoretical knowledge. A second paper on the shrimp followed in 1892†, and this deals more closely with the correlations. Weldon now replaces medians by means in both marginal totals and arrays. He still uses probable errors or quartiles, and goes through the laborious process of reducing each deviation to the probable error. He uses r "in accordance with Mr Galton's notation" to represent the constant which measures the "degree of correlation" between organs. I think, but it is not quite clear, that he determined his probable error from the mean error, not from the quartile. He then determined r from each individual array and took the mean value of these r's as the true r. He accordingly introduced a greatly increased accuracy into the computing of correlation. He dealt with five local races of shrimps and found correlations for 22 pairs of organs. His regression diagram, p. 8, is still an admirable sample of this type of work. The correlations between post-spinous portion and total carapace length may be cited as illustrations of what Weldon and Galton were testing:

Plymouth (1000)	$r = 0{\cdot}81$
Southport (800)	$r = 0{\cdot}85$
Roscoff (500)	$r = 0{\cdot}80$
Sheerness (380)	$r = 0{\cdot}85$
Helder (300)	$r = 0{\cdot}83$

The suggestion that r has the same value for all races of the same species was supposed to be confirmed by these results. We now realise that without a knowledge of the probable error of r, such a statement is illusory. But it was this very series of values which led to the investigation of the probable error of r and so to the extension of the correlational calculus.

* *R. S. Proc.* Vol. XLVII. p. 445, 1890.

† *R. S. Proc.* Vol. XLI. p. 2.

In this paper Weldon also published for the first time with due appreciation of their meaning *negative* correlation coefficients. In conclusion Weldon remarks: "A large series of such specific constants would give an altogether new kind of knowledge of the physiological connexion between the various organs of animals; while a study of those relations which remain constant through large groups of species would give an idea, attainable at present in no other way, of the functional correlations between various organs which have led to the establishment of the great subdivisions of the animal kingdom" (p. 11). In these lines we can read the starting-point of biometry as applied to other types of life than man.

I will not keep you longer over Weldon's contributions than to say that in 1893 appeared his third statistical paper* on "Correlated Variations in Naples and Plymouth shore Crabs." Weldon dealt with 23 pairs of organs in both Naples and Plymouth races. He proposes to call r "Galton's function"†. The paper shows that the 23 values of r at Plymouth and Naples are fairly close, but was again inconclusive because the significance of the differences could not be ascertained without a knowledge of the probable error of r.

We may next turn to Edgeworth, whose fundamental paper is that on "Correlated Averages" which appeared in the *Philosophical Magazine* of August, 1892, pp. 190—204. Edgeworth starts by referring to Galton's memoir of 1888 and Weldon's of 1892 on shrimps. He assumes for the probability that any particular values $x_1, x_2, \ldots$ shall occur

$$\Pi = Je^{-R} dx_1 dx_2 dx_3 \ldots,$$

where R is

$$= p_1 (x_1 - \bar{x}_1)^2 + p_2 (x_2 - \bar{x}_2)^2 + \ldots + 2q_{12} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + \ldots.$$

He does not justify this assumption but hopes to do so in a subsequent paper. He states that Galton by the happy device of measuring each deviation by the corresponding quartile had reduced in the case of two variates

$$R = \frac{x_1^2}{1-\rho^2} - \frac{2\rho x_1 x_2}{1-\rho^2} + \frac{x_2^2}{1-\rho^2}$$

to the discovery of a single constant ρ. This is hardly accurate; to reduce the expression R to the above it would be needful to measure not in terms of the quartile but of $\sqrt{2}$ S.D., which is I think sometimes termed the 'modulus‡. Edgeworth replaces Galton's "Index of Co-relation" and Weldon's "Galton's Function" by the term "coefficient of correlation." He then proceeds to weaken down Weldon's process of finding a mean r by suggesting that it will be adequate to find it by taking some of the ratios of 'subject' and mean 'relative' instead of the whole series. I look upon this suggestion as a distinctly retrogressive step.

* *R. S. Proc.* Vol. LIV. pp. 318—329.

† "The importance of this constant in all attempts to deal with the problems of animal variation was first pointed out by Mr Galton...and I would suggest that the constant whose changes he has investigated and whose importance he has indicated, may fitly be known as 'Galton's function,'" p. 325.

‡ Edgeworth appears to realise this on p. 194, but he did not go back and correct his statement of p. 190.

Our object should be to find the 'best value' for r and not how it may be most easily determined at the obvious cost of accuracy.

Although I am unable to follow some of Edgeworth's notation, he undoubtedly reaches something like the correct value for the correlation surface of three variates. In his notation

$$R = \Delta\{(1-\rho_{23}^2)x_1^2 + (1-\rho_{31}^2)x_2^2 + (1-\rho_{12}^2)x_3^2 - 2x_1x_2(\rho_{12}-\rho_{13}\rho_{23}) - 2x_2x_3(\rho_{23}-\rho_{21}\rho_{31}) - 2x_3x_1(\rho_{31}-\rho_{32}\rho_{12})\},$$

where

$$\Delta^{-1} = \{(1-\rho_{13}^2)(1-\rho_{12}^2) - (\rho_{12}\rho_{23}-\rho_{13})^2\}$$

according to him, but the factor $1-\rho_{13}^2$ should be replaced by $1-\rho_{23}^2$ I think.

Even with this change I am unable to reach the value he gives in Galton's case of

$$\rho_{12} = \cdot 8, \quad \rho_{13} = \cdot 9, \quad \rho_{23} = \cdot 8,$$

for these seem to give

$$\Delta = 9{\cdot}9305,$$

whereas Edgeworth's value is 16·129.

I do not grasp his equation at the foot of p. 196, nor follow how the equation at the top of p. 197 follows from it.

Lastly we come to p. 201 where we should expect to find the general regression equation. Edgeworth tells us that the reasoning is quite general and accordingly we ought to anticipate that his results whatever they are would give our accepted values

$$p_{ss} = \frac{R_{ss}}{R} \text{ and } p_{ss'} = \frac{R_{ss'}}{R},$$

where R is the determinant of the correlations. Instead of this simple rule Edgeworth sums up in the middle of the page with equations

$$\begin{aligned} \Delta\rho_{13} &= +\Delta^3(\rho_{24}\rho_{21}\rho_{42}), \\ \Delta\rho_{14} &= -\Delta^3(\rho_{21}\rho_{32}\rho_{43}), \\ &\vdots \quad\quad \vdots \quad\quad \vdots \end{aligned}$$

There is no explanation of what the symbolism means, and I cannot interpret it, so as to provide the requisite generalisation for n variates.

On the other hand while unable to interpret Edgeworth's general analysis I agree in the case of four variates with the only two terms I have taken the trouble to test in his numerical illustration of this case,

$$\rho_{12} = \frac{1}{\sqrt{2}}, \quad \rho_{13} = \sqrt{\frac{1}{3}}, \quad \rho_{14} = \sqrt{\frac{1}{4}}, \quad \rho_{23} = \sqrt{\frac{2}{3}}, \quad \rho_{24} = \sqrt{\frac{2}{4}}, \quad \rho_{34} = \sqrt{\frac{3}{4}},$$

namely 2 as the coefficient of x_1^2 and $-2\sqrt{2}$ as the coefficient of x_1x_2, my R being $\frac{1}{24}$, R_{11} being $\frac{1}{12}$ and R_{12} being $\frac{-\sqrt{2}}{24}$. Edgeworth does not provide the needful external constant of the frequency surface, i.e.

$$\frac{N}{(2\pi)^n\sigma_1\sigma_2\ldots\sigma_n}\frac{1}{\sqrt{R}}.$$

I should sum up Edgeworth's work of 1892 by saying that he left the problem of multiple correlation at least in a very incomplete state. He probably knew what he was seeking himself, but he did not give the requisite attention to the wording or printing of his memoir to make it clear to others, and accordingly in looking back at the matter now I am very doubtful whether in 1895 I ought to have called the problem of multiple correlation, "Edgeworth's Problem." He certainly did not put the answer to it in a form in which the statistician with a customary amount of mathematical training could determine the form of the surface for n variates, as soon as their S.D.'s and correlations had been calculated. I think I am justified in saying this for I have not to my recollection come across any treatment of multiple correlation which starts from Edgeworth's paper or uses his notation.

It will be seen from what has gone before that in 1892 the next steps to be taken were clearly indicated. They were, I think,

(*a*) The abolition of the median and quartile processes as too inexact for accurate statistics.

(*b*) The replacement of the laborious processes of dividing by the quartiles and averaging the deduced values of r, by a direct and if possible 'best' method of finding r.

(*c*) The determination of the probable errors of r as found by the 'best' and other methods.

(*d*) The expression of the multiple correlation surface in an adequate and simple form.

These problems were solved by Dr Sheppard or myself before the end of 1897.

Closely associated with these problems arose the question of generalising correlation. Why should the distribution be Gaussian, why should the regression curve be linear?

As early as 1893 I dealt with quite a number of correlation tables for long series and was able to demonstrate

(i) by applying Galton's process of drawing contours of equal frequency that most smooth and definite systems of contours can arise from long series, obviously mathematical families of curves, which are (*a*) ovaloid, not ellipsoid, and (*b*) which do not possess—like the normal surface contours—more than one axis of symmetry,

(ii) that regression curves can be quite smooth mathematical curves differing widely from straight lines,

(iii) that in cases wherein (i) and (ii) hold, homoscedasticity is not the rule.

I obtained differential equations to such systems, but for more than 25 years while often returning to them, have failed to obtain their integration.

This seems to me the desideratum of the theory of correlation at the present time: the discovery of an appropriate system of surfaces, which will give bi-variate skew frequency. We want to free ourselves from the limitations of the normal surface, as we have from the normal curve of errors.

As early as 1897 Mr G. U. Yule*, then my assistant, made an attempt in this direction. He fitted a line or plane by the method of least squares to a swarm of points, and this has been extended later to n-variates and is one of the best ways of reaching the multiple regression equations and the coefficient of multiple correlation†. Now while these methods are convenient or utile, we may gravely doubt whether they are more accurate *theoretically* than the assumption of a normal distribution. Are we not making a fetish of the method of least squares as others made a fetish of the normal distribution? For how shall we determine that we are getting a 'best fit' to our system by the method of least squares?

If we are fitting a curve $y = f(x, c_1, c_2, c_3)$

to a series of observations we can only assert that least square methods are theoretically accurate on the assumption that our observations of y for a given x obey the normal law of errors. That is the proof which Gauss gave of his method and I personally know no other. *Theoretically* therefore to have justification for using the method of least squares to fit a line or plane to a swarm of points we must assume the arrays to follow a normal distribution. If they do not, we may defend least squares as likely to give a fairly good result but we cannot demonstrate its accuracy. Hence in disregarding normal distributions and claiming great generality for our correlation by merely using the principle of least squares, we are really depriving that principle of the basis of its theoretical accuracy, and the apparent generalisation has been gained merely at the expense of theoretical validity. Take other distributions of deviations for the arrays and the method of least squares is not the one which will naturally arise from making the combined probability a maximum. It is by no means clear therefore that Mr Yule's generalisation indicates the real line of future advance.

I have endeavoured to indicate in this paper the broad outline of the early history of correlation which has now a most extensive literature. It is a long step from Francis Galton's 'reversion' in sweet pea seeds to the full theory of multiple correlation, which we now know to be identical with the spherical trigonometry of high-dimensioned space, the total correlation coefficients being the cosines of the edges of the polyhedra and the partial correlation coefficients the cosines of the polyhedral angles. But to find the correlation of the health of a child with the number of people per room while you render neutral its age, the health of its parents, the wages of its father, and the habits of its mother, is no less vital a problem than Galton's correlation of character in parent and offspring. It requires indeed more mathematics, but the mathematics are not there for the joy of the analyst but because they are essential to the solution. It is the transition from the mill as pestle and mortar to the mill with steam driven grain crushing steel rollers. But the inventor of milling was the person who bruised grain between two stones, and Galton was the man who discovered the highway across this new country with what he aptly terms "its easy descents to different goals."

* *Journal of Royal Statistical Society*, Vol. LX. Part iv, p. 3.

† *Biometrika*, Vol. VIII. p. 438. The method adopted in the paper is not that of fitting a generalised plane by least squares, but of making a generalised correlation coefficient take its maximum value. It appeals only to the rules of the differential calculus and not to the method of least squares, or indirectly to Gauss' law of errors.

HISTOLOGICAL EXAMINATION OF AN ADULT HUMAN ALBINO'S EYEBALL, WITH A NOTE ON MESOBLASTIC PIGMENTATION IN FOETAL EYES.

By C. H. USHER, M.B., B.C., Camb.

A. Introductory.

When some ten years ago the subject of albinism was taken up by Karl Pearson, the late E. Nettleship and myself, the following problems naturally arose from the then state of our knowledge of the subject.

(i) Is there or is there not a complete congenital absence of pigment in the eye of the albino? This problem follows at once on the definition which had several times been repeated, that an albino was a person with complete congenital absence of pigment in the eye. The answer to this first problem could not be reached by clinical methods but only by microscopical examination of excised eyes. At the time just referred to not a single human eye had been reported upon which was entirely devoid of pigment. An examination of a portion of the iris only, even if it is found pigmentless, cannot be conclusive for settling the question of entire absence.

(ii) The presence or absence of a fovea and of yellow colour at the macula in the case of the albinotic eye. Presence and absence of both have been asserted. But in some of the few available cases there is no evidence that a thorough microscopic examination of an adequate series of sections was made.

(iii) What truth is there in the theory originally propounded by Meckel and Mansfeld, that the source of albinism is to be sought in an arrest of development? Such a theory involves the extent of pigmentation in an albino corresponding to that of some foetal stage. Accordingly the study of the development of pigmentation in the foetal eye becomes of very great interest. It would be of greater interest still, if we knew or could find out anything about the pigmentation in the eye of the albinotic foetus.

The accompanying study is a contribution of data bearing upon the above problems. It consists in the microscopic examination of an albinotic eye and of the eyes of normal dark-raced foetuses.

B. The Eye of the Adult Human Albino.

In literature there are few records of a microscopical examination of a human albino eye. In all, these relate to two pieces of iris, removed during operations for the extraction of senile cataract(6), and to the eyeballs of six other cases* (Manz, Nettleship, Fritsch, Adler and McIntosh, Coats, Elschnig). Of the two pieces of iris, one (Nettleship, Fig. F) contained no pigment, the other (Usher) had an abnormally small quantity of pigment in the two posterior epithelial layers and not any in the stroma (Fig. D). Of the six albinos where the whole eyeball was obtained, five were adults, and the sixth was an infant of ten weeks. All of the eyes from four of the adults contained some pigment, in the fifth case Fritsch(7) makes no mention of either the presence or absence of pigment in the eye. In the case of Manz(3) and Elschnig(16), there was some pigment present in the retinal epithelium, but none at all in the stroma of the iris and ciliary body or in the choroid. In the case described by Nettleship(6), the retinal epithelium was pigmented, but the pigment was in much less quantity than normally. There was some pigment in the choroid, more particularly at the macular region, but the stroma of iris and ciliary body was without pigment (Fig. C). Coats(15), in the eye of an albinotic negro (Fig. E), found that the iris pigment epithelium was brown, considerably lighter than in the normal blue European eye. The epithelial pigmentation of the ciliary body was as deep as in many European eyes. The pigmentation was very light in the whole of the posterior part of the globe. The iris stroma was pigmented, but not the stroma of the ciliary body or the choroid (Fig. E). In the infant's eyes, Adler and McIntosh(11) found no pigment. They make no special mention of the histological appearances of the macula. For two reasons an account of this region would have been of interest. In the first place no one has so far recorded the presence of a normal fovea in a human albino eyeball—true, in this albino baby a fovea is mentioned as having been seen macroscopically—and in the second place there is reason to believe that it is to this region particular attention should be given in a search for pigment, and that no eye should be pronounced free from pigment until this part has been carefully examined. It is well known that in the normal European eye the hexagonal cells of the retina are darker at the macula than in the parts immediately surrounding it. Also, the choroidal pigment is frequently dense in this region. Again, in an eye of one of the five adult albinos there is a small quantity of choroidal pigment principally located at the macula, and in the case to be described the only mesoblastic pigment present is situated at the macular region. Furthermore, from an examination of eyes from twelve dark-race foetuses, and newborn and very young infants (see below, pp. 51—4), I have found pigment present considerably before birth both in the iris stroma, and in the choroid especially at

* Since this paper was written I have found in the *Muenchener Medizinische Wochenschrift*, Nr. 26, 26 June 1917, S. 845, Velhagen's account of a microscopical examination of the eyes of an albinotic man, age 23. No pigment was present in choroid, stroma of iris, and retinal epithelial layer as far as ora serrata. The ciliary body epithelium and posterior surface of iris were pigmented. The macula lutea was not differentiated, no fovea was recognised.

the posterior part of the eye. In a five-months' Malay-Tamil foetus pigment was present in the choroid at the posterior part of the eye (Fig. 1) when no pigment could be detected in the iris stroma. At the same region of the choroid, pigment was present in a seven-months' Cantonese foetus, whilst in the iris it was only with the aid of an oil immersion lens that a few chromatophores containing faint brown granules could be found. In a seven-months' Tamil foetus, the choroid contained round brown pigment granules in the bodies and processes of the chromatophores (Fig. 2). At birth there is much pigment in the iris stroma (Fig. 3) and choroid of these dark-raced eyes, but it falls far short of what obtains in the adult Tamil (Figs. 4 and 5), Chinese, and some other dark-race eyes. Therefore, there is increase of pigment in these structures after birth. In view of these observations and from what is known of the distribution of ocular pigment in albinotic lower animals, it is conceivable that a human albinotic eye with scanty pigmentation may yet be found in which the pigment is limited to the macular region and located in the retinal hexagonal cells and the choroidal cells, or else confined to either the one or the other of these structures. No case has so far been published of a human eyeball containing pigment in the choroid with entire absence of pigment in the hexagonal cells of the retina, though such a condition we have found in the Avian eye—that of a hedge sparrow (*Accentor modularis*). In the iris of man clinical examination, in a few cases, has shown that pigment may be absent from the epiblastic layers whilst present in the stroma (17), though a microscopical examination is still required in such cases to determine whether the pigment is completely absent from the posterior layers. It is probable, as Adler and McIntosh suggest in their case, that had she lived, this infant of ten weeks would have remained a complete albino. In normal European infants, however, the iris, stroma and choroid contain little if any pigment at birth*, and for some months afterwards (18).

The following are the notes of the case before us:

Alexina H., female, age 17, native of Banffshire, admitted to the Aberdeen Royal Infirmary on November 15th, 1918, under the care of Professor A. W. Mackintosh, to whom I am indebted for the particulars of her illness. Since the age of 12 she has been subject to epileptic fits†. In February 1918 she had rheumatic fever, and since that time has never been strong. The diagnosis

* Rieke (4) states that the earliest deposition of pigment in the human choroid is in the seventh month of foetal life, but that this is by no means the case in all individuals as there is great variation.

Schreiber and Schneider (8) found no trace of pigment in the stroma cells of the uvea in a thirty-three weeks' foetus.

Lauber (9) examined the eyes of human embryos, and found that stroma pigment was completely absent in the eighth month. In an early-born foetus of the ninth month a pigmentation of the stroma cells of the iris could be made out, whilst the choroid was still completely pigmentless.

Treacher Collins and Mayou (12) give the seventh month as the average time at which pigment first makes its appearance in the branched cells of the choroid. The pigmentation of the stroma cells of the iris, they state, does not commence until after birth.

v. Szily (13) says that the pigment of mesodermal origin appears in man shortly before birth or even later, and that it is fully developed during the first year.

† She also had fits in infancy.

made was epilepsy, meningitis, and mitral incompetence. Her thyroid was definitely enlarged; knee jerks plus; plantar reflexes not elicited; no ankle clonus; difficulty in articulation suggestive of a bulbar lesion; reported difficulty in swallowing; patient now cannot walk. On November 18th had incontinence of urine and faeces. On November 20th lumbar puncture was made. The fluid was clear and under slight pressure. Professor Shennan's report of the cerebro-spinal fluid states that "Few gram-negative cocci which did not grow in culture medium were found present." The temperature gradually rose from normal on November 17th to 105° on November 22nd, when she died.

On November 18th I noted that the optic discs were normal. The choroidal vessels were markedly exposed except at the macular region. Each fundus was albinotic. No red reflex was visible in the pupils. The iris was blue, and no pigment could be detected in the stroma, even when it was viewed through a loupe. There was constant lateral nystagmus in each eye. The skin was blond, and the hair was pale yellow in artificial light. When seen again three days later, in good day-light, she was semi-conscious, and at first the eyeballs were motionless, but on being roused a little, the nystagmus commenced again though the movements were slow. In this light the hair was tow colour. When attending the Ophthalmic Department of the Infirmary on October 5th, 1911, the refraction of the eyes was estimated after the use of atropine. The right eye had 1·5 D. of hypermetropia in an oblique meridian and 3 D. of myopia in the opposite meridian; acuteness of vision with correction was $\frac{6}{36}$. The left eye had 1·5 D. of hypermetropia in one meridian, and 2·25 D. of myopia in the opposite meridian; acuteness of vision with correction was $\frac{6}{36}$. At the same date the pupils looked black, though occasionally a port-wine colour was visible. A description of this case, by Dr W. C. Souter, and the pedigree, is published in *Albinism in Man*, Pt. IV. No. 646. At this date, 1909, the hair was white and contained no granules and no diffused pigment. The eyes were less myopic and the pupils were red. No general post-mortem examination was obtained. The left eye was excised 17 hours after death and placed in Zenker's fluid. The eye was opened equatorially, embedded in celloidin and then sectioned.

Macroscopically the ciliary body was well pigmented, and definite brown pigmentation of the fundus was visible. A horizontal fold of retina passed outwards from the optic disc through the macula, and a smaller fold passed nasalwards from the optic disc. (Folds of the retina at the macula are of common occurrence in eyes removed after death.) No fovea was discernible. The retina at the macular region was of a pale yellow colour, such as is seen in the normal eye. A complete series of consecutive sections, cut horizontally, was made through the macula. No section was lost. The anterior portion of the eyeball was cut antero-posteriorly.

Microscopical Examination. The iris contains no stroma pigment. Sections both stained and unstained were searched for pigment with an oil immersion lens. The two epiblastic layers at the back of the iris are pigmented, but not so deeply

as those of a normal blue or brown eye. The nuclei of a number of the cells are clearly visible (Fig. A), whilst in a normal eye (Fig. B) they cannot be distinguished because of the dense pigmentation. The pigmented derivatives of the epiblastic layers, V. Michel's and Fuch's spurs at the region of the sphincter muscle, and Grunert's spur at the root of the iris, are feebly represented, and no round clump cells are present. The ciliary body epithelial pigment layer contains less pigment than that in a normal eye, and the nuclei of some of its cells are visible. This layer contains, however, much pigment and terminates posteriorly at the ora serrata in the feebly pigmented hexagonal cells of the retina. The stroma of the ciliary body contains no pigment. The choroid is devoid of pigment, except at the macular region, where there are a few pigmented chromatophores. The hexagonal cells of the retina are considerably less pigmented than those in a normal eye. They are darker at the macula than in the region surrounding it. They contain elongated pigment granules. The retina has several layers of ganglion cells at the macula as in normal eyes, but in none of the sections is there a fovea visible*. The nuclear layers are both present in every section and neither of them disappears at any point, and the same holds good for the ganglion cell layers. The elements composing the bacillary layer have undergone post-mortem change, which makes it impossible to state what their condition was during life.

The distribution of pigment then in this eye as regards the retinal epithelium, is the same as in a normal eye, but it is less in quantity. Mesoblastic pigmentation is entirely absent in the iris, ciliary body and most of the choroid, only a few cells at the macular region contain any pigment. In all these respects this eye resembles the case published by Nettleship (6), in which, however, there was rather more choroidal pigment, some of it lying in the parts of the choroid away from the macular region. It resembles it also in that no proper fovea is found in the sections. Fritsch (7) found the fovea imperfectly developed in the retina of an albinotic Herero. And in Elschnig's case there was no fovea, rods as well as cones were present at the macula, and the hexagonal cells contained no needle-shaped granules.

These four cases appear to be the only ones in albinos in which the fovea was looked for microscopically, and in none of them was it found fully developed. Absence of the fovea has been recorded by Seefelder (10) in a case of aniridia with nystagmus. In a case of hereditary nystagmus (14), seen by myself, in a child aged 2 years, with blond skin, yellow brown hair, lashes and eyebrows, and with constant lateral nystagmus, no proper fovea could be found in either eye. The retinal epithelium was normally pigmented throughout, and pigmented chromatophores were present in the choroid and in small numbers in the iris. Since Fritsch and Elschnig published their cases, Ichikawa (19), with Elschnig, examined the macular region of six albinos with the ophthalmoscope and found that there was

* A puzzling appearance seen in a few consecutive sections can be explained by a folding of the retina, and it is difficult to conceive how a normal fovea could escape detection even at this part.

no differentiation, corresponding to the appearance of the macula and fovea in a normal eye, and that the arrangement of the retinal blood vessels at this region was different from that seen in normal eyes, in that instead of converging regularly towards a fovea, they met and crossed each other irregularly.

When a normal human eyeball is opened and examined macroscopically, a diffuse pale yellow colour is visible in the retina at the macula. In albino eyes examined by Manz, Fritsch and Carron du Villards (2), no yellow was found at the macula, while in a case by Buzzi (1), and in the one already referred to by Nettleship, the yellow colour was present, as in this case. During life the yellow colour at the macula is not seen with the ophthalmoscope, when the usual sources of illumination are employed. Recently however Affolter (20), by using red free light, has been able to see the yellow colouration with the ophthalmoscope, but he failed to find it in two albinos' eyes.

The imperfect vision in albinos has been explained by attributing it principally to the absence of pigment, or defective pigmentation, in the retinal epithelium, but the results obtained by histological examination of the macula in these few human albino eyes, and the records of the two cases with nystagmus, already referred to (aniridia and hereditary nystagmus), in which the fovea was absent, suggest another explanation, and pending further investigation of the macular region in the human albino eye, there is reason to regard the imperfectly developed or absent fovea as possibly the chief cause of the defective vision and nystagmus in albinos.

C. On the Ocular Mesoblastic Pigmentation of the Iris and Choroid in some Dark-Race Eyes.

The following eyes were examined:

1. Tamil seven-months' foetus. Eyes received from Dr H. Fraser, from the Federated Malay States (Kuala Lumpur), November 30th, 1915. Iris stroma contains some pigment, even with a low magnification it is evident that the anterior layers of the stroma are lightly pigmented, and with high powers numerous brown granules are seen in both bodies and processes of the chromatophores. None of the large round dark cells so conspicuous in the adult dark-race iris are present. Ciliary body contains lightly pigmented chromatophores. Choroid has well pigmented cells at macular region. Further forwards the cells are not so well pigmented.

2. Tamil seven-months' foetus. Eyes received from Dr H. Fraser, Federated Malay States (Kuala Lumpur), November 30th, 1915. The eyes are smaller than those of No. 1. In the iris stroma, only a few chromatophores are found containing pigment granules, and a few are present in the ciliary body. The choroid contains brown, round, pigment granules in the bodies and processes of the chromatophores, especially well marked at the macula (Fig. 2). None of the cells are so dark as those in an adult Tamil eye, and they become very lightly pigmented away from the macular region. The pigmentation is much greater in the choroid than in the iris.

3. Tamil full-time foetus. Eyes received in alcohol, after formaline, from Dr H. Fraser, Kuala Lumpur, Federated Malay States, May 10th, 1910. Antero-posterior and equatorial diameters of the eyes are both equal to 18 mm. Iris stroma is lightly pigmented. Pigment is most marked at the anterior surface of the iris. The branched cells are not so dark as in the adult Tamil's eye. Choroid contains many pigmented cells, but these are neither so numerous or quite so dark as in the adult Tamil eye. Choroid is darker at yellow spot than elsewhere.

4. Tamil foetal eyes received in spirit, after formaline, from Dr H. Fraser, Kuala Lumpur, Federated Malay States, March 10th, 1910. Eye measurements: right eye, antero-posterior diameter = 19·5 mm., equatorial diameter = 18 mm.; left eye in both diameters is equal to 18·5 mm. Iris contains much stroma pigment, which is most marked anteriorly and around blood vessels. No clump cells found. In front of the sphincter muscle the cells are branched. Ciliary body stroma is pigmented, but much less densely than in the adult. Choroid is pigmented most markedly at its outer parts and around blood vessels.

5. Tamil full-time foetus. Both parents are Tamils. Eyes received in 10% formaline, from Dr H. Fraser, Kuala Lumpur, Federated Malay States, October 21st, 1915. Antero-posterior diameter of each eye = 18·1 mm., equatorial diameter of each eye = 17·7 mm. The chromatophores in the iris stroma are well defined and contain brown pigment granules. As usual the pigmentation is most marked in the anterior layers of the iris which are heavily pigmented (Fig 3). Ciliary body has pigmented chromatophores. The choroid contains many pigmented cells.

6. Malay-Tamil foetus (Father a Malay, Mother a Tamil), age 5—6 months. Eyes received in formaline from Dr George A. Finlayson, Singapore, July 13th, 1910. Both antero-posterior and equatorial diameters measure 9 mm. The iris and the ciliary body have no pigment in the stroma. The choroid contains a few branched cells with pigment granules (Fig. 1). These cells are fewer than those in a seven-months' Chinese foetal eye, No. 10, and are principally situated at the posterior part of the eye.

7. Dark-race infant (probably Malay, Chinese, or Tamil) thought to be still-born. Eyes received in formaline from the late Dr Robert D. Keith, Singapore, March 1st, 1915. Antero-posterior diameter of each eye = 18 mm. The iris chromatophores are pigmented with small brown granules that lie in the cell bodies and processes, some cells are more pigmented than others. The ciliary body has pigmented chromatophores as in the iris. The choroid is rather heavily pigmented, especially at its outer part, and at the posterior pole of the eye.

8 and 9. Two Chinese infants. The age of one is "about 2 weeks"; of the other, not stated. Eyes received from Dr George A. Finlayson, Singapore, March 14th, 1914. As the distribution and quantity of pigment in the iris and choroid is similar in these two cases, one description will serve for both. The iris stroma cells contain brown pigment granules. The amount of pigment in the

individual cells varies considerably. At the anterior part of the iris the chromatophores are packed together, giving a darker appearance to this part of the iris. No dark, round, unbranched cells are present, which are so numerous in the irides of adult Tamils and Chinese. (There are a few dark brown ones present in one case.) The ciliary body contains lightly pigmented chromatophores. There is much pigment in the choroid lying in branched cells, especially posteriorly.

10. Chinese foetus (Canton) "about 7th month, perhaps a little older." Eyes received in formaline from Dr George A. Finlayson, Singapore, January 15th, 1910. Eye measurements: antero-posterior diameter about 16 mm., equatorial diameter = 14 mm. The iris stroma is unpigmented, with the exception of a few pale brown chromatophores at the anterior surface of the iris, which contain pigment granules. In the choroid is a considerable quantity of pigment, lying in elongated and in branched cells, though very much less than in the full-time foetus No. 3, Tamil foetus No. 4, and in a negro foetus No. 11. The pigment is most marked in the outer layers of the choroid, and there is some around the blood vessels.

11. Fantis Negro, full-time twin foetus. Eyes received in formaline from Dr F. W. Moir, through Dr Alexander Low, February 7th, 1910. Eye measurements: both antero-posterior and equatorial diameters = 18 mm. The iris stroma contains a quantity of pigment, though much less than in an adult dark-race eye; there are comparatively few large pigmented branched cells; especially in anterior layers of iris and around the blood vessels are numerous round black cells. Many of these round cells show no evidence of branches, but on high magnification many of them, though roughly circular, exhibit different forms. In the ciliary body are comparatively few pigmented chromatophores. The choroid contains a considerable quantity of pigment in branched and elongated cells; many of the cells are only lightly pigmented, with brown pigment granules. The pigmentation is not nearly so dense as in a dark-race adult eye.

12. Tamil infant. Eyes received from the late Dr Robert D. Keith, Singapore, on July 29th, 1913. Eye measurements: antero-posterior and equatorial diameters are both equal to 18 mm. The iris contains much brown pigment in the branched chromatophores and some large, round, dark cells are present, but not in distinct groups. The epiblastic layers are dark and show well-marked spurs. The ciliary body contains many pigmented chromatophores. The choroid is well pigmented, and much pigment is present round the optic nerve entrance. Neither iris nor choroid are as deeply pigmented as those of an adult Tamil eye.

The pigmented retinal epithelium on the posterior surface of the iris, in ciliary body, and on the inner surface of the choroid, is exceedingly dark in every one of these twelve cases. This is to be expected, as the retinal pigment is fully formed by the twelfth week of foetal life (18). It is interesting to find mesoblastic pigmentation so early as is seen in some of these cases. The age of the foetus or infant has usually been given by the medical man who sent its eyes, and in most cases, measurements of the eyeball were made before opening. In case No. 6 where

pigmented chromatophores are present in the choroid, the age of the foetus is given as 5—6 months and the eye measurements were 9 mm. E. Treacher Collins (5) measured the eye-balls of some human foetuses and found that at the fourth month, two eye-balls measured 8·5 mm. in the antero-posterior diameter, 8 mm. in the vertical, and 8 mm. in the lateral diameter. At the sixth month, two eye-balls measured 10·3 mm. in the antero-posterior diameter, 9·75 mm. in the vertical, and 10 mm. in the lateral diameter. According to these measurements, case No. 6 would certainly not be older than 5 months. It is quite obvious from an examination of these cases that the mesoblastic pigment in the eyes of dark races appears earlier and is laid down in foetal life in greater quantity than in European eyes. These notes refer only to the pigmentation of the choroid, ciliary body and iris, but mesoblastic pigment is conspicuous in some of the cases at other parts of the eye, such as along the course of the ciliary blood vessels in the sclerotic.

D. General Conclusions.

(*a*) Of the six albino eyeballs thus far examined, the albinos being determined clinically by the usual clinical characters, four are found to have pigment, of a fifth no statement was made by the examiner as to pigment. The sixth case was that of an *infant*, and here no trace of pigment was found in any of the structures of the eye. The remaining examinations of the human albino eye are concerned only with portions of the iris; in one no pigment, in the other some pigment, was observed. The former case provides no evidence as to whether the whole eyeball was or was not without pigment. We must conclude therefore that the total absence of pigment in the eye cannot be used as a definition of human albinism.

(*b*) As apart from absence of pigment, the absence or imperfect development of the *fovea centralis* shown to occur in albinotic eyes may possibly be the chief cause of defective vision in these cases.

(*c*) An examination has been made of the distribution of pigment in the eyes of a number of dark-raced individuals, adult, infant and foetal. The notes made indicate that in the dark-race eye mesoblastic pigment appears earlier in foetal life, especially in the choroid, and is in much larger quantity by the time of birth than in the European eye.

According to this we should expect, if albinism consists of "an arrest of development," more pigment in albinos of dark races than in albinos of white races. Clinical examination of such albinos appears to indicate that this is a fact, but it is a fact which might be accounted for on other than the Meckel-Mansfeld hypothesis. At the same time it renders any definition of human albinism as a complete absence of ocular pigment less valid for any practical service in the study of the heredity of albinism.

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DESCRIPTION OF PLATES.

PLATE I.

Fig. 1. Section through choroid and retinal pigment layer of a dark-race foetus, age 5—6 months. $\times 600 \times \frac{2}{3}$. Shows branched cells with pigment granules.

Fig. 2. Section through choroid and retinal pigment layer of a seven-months' Tamil foetus. $\times 600 \times \frac{2}{3}$. Shows pigmented chromatophores.

Fig. 3. Section of iris from full-time Tamil foetus. $\times 125 \times \frac{2}{3}$.

Fig. 4. Section of iris from adult Tamil, age 27. $\times 125 \times \frac{2}{3}$.

Fig. 5. Section through choroid and retinal pigment layer of adult Tamil, age 27. $\times 600 \times \frac{2}{3}$. (Compare with Figs. 1 and 2.)

PLATE II.

Fig. A. Section of iris from British female albino, age 17. $\times 105 \times \frac{2}{3}$. Nuclei of some epiblastic pigment cells are visible.

Fig. B. Section of a normal British iris. $\times 105 \times \frac{2}{3}$.

Fig. C. Section of iris from British male albino, age 67. $\times 105 \times \frac{2}{3}$. Brown pigment is present in both layers of the epithelium on the back of the iris.

Fig. D. Section of iris from British male albino, age 74. $\times 330 \times \frac{2}{3}$. A bit of iris was removed at an extraction of cataract. Pigment is present in the cells of both epithelial layers on the back of the iris.

Fig. E. Section of iris from Negro albino. $\times 105 \times \frac{2}{3}$. (Coat's case.) Shows pigmented chromatophores as well as brown pigment in posterior epithelial layers.

Fig. F. Section of iris of a remarkably perfect British male albino, age 74. Portion of iris removed during the operation for extraction of cataract. (Nettleship's case.) Iris is pigmentless.

The original drawings were reduced 7 : 6 for the plate reproduction.

ON THE INHERITANCE OF THE FINGER-PRINT.

By ETHEL M. ELDERTON, Galton Research Fellow.

FIRST PAPER.

(1) *Explanatory and Historical.*

The inheritance of finger-print types when once those types have been analysed and classified seems an almost ideal subject for the study of heredity in man. It would appear at first sight that they would present most excellent material for comparing various theories of inheritance and measuring the intensity of heredity between the different grades of kinship. Yet as far as I am aware nothing has hitherto been published on the subject either from the Biometric or the Mendelian standpoint. By 1903, data had been collected and reduced showing the degrees of inheritance for many physical characters in both animals and man and the collection and reduction of data for the heredity of mental and moral characteristics in man had been started. It would appear strange that finger-print types should remain unregarded,—the "arch," the "loop," the "whorl," the "composite" were familiar to many and seemed by their easy determination for the purpose of criminal indices to be most appropriate for the study of heredity. Two chief difficulties, however, stood in the way. First the great labour of collecting in adequate quantities the finger-prints of relatives; and secondly the still more important fact that while four or five broad categories were adequate for a criminal index, they were inadequate for scientific distinctions. Galton at first ran up his four or five categories to 53, and the continual appearance of transitional forms led him further and further in the sense of finger-print continuity. Without exaggeration one can say that he sought for years for quantitative measures of the finger-print*, which might be applied to any type, and that, when he had appreciated continuity, it was this failure to obtain quantitative measurement which ultimately led him to put the problem on one side. Nevertheless those who were intimate with him in the last ten years of his life know the prominent part which the inheritance of finger-print types played both in his thoughts and activities.

* The Galton Laboratory possesses a great variety of cameras and other apparatus for enlarging, orientating and measuring finger-prints designed by Galton. Many of these, especially those for angular measurements on the finger-print, are now, failing any memorandum of purpose, hardly interpretable.

In 1903* Galton set about obtaining a collection of family finger-prints for the purpose of studying heredity. The persistence of finger-print type throughout life established by Herschel and Galton was to make the inquiry a simple one; it was unnecessary to collect adults only; classification could take place at any time during life.

As early as 1892 Galton had made a preliminary inquiry into the inheritance of finger-print types, but he was not satisfied with either the extent of his material, or with the classification of types, which he had at that time evolved.

We shall give the results he obtained below. They were published in 1892 in his work entitled *Finger-Prints.*

In this book Galton devotes a chapter to the question of inheritance but at that time there was no method of obtaining a correlation coefficient when the characters were not quantitative and Galton could only show that type of finger-print is inherited but could not measure the strength of inheritance nor compare it with the results obtained from the study of other characters. The method Galton used was the following: he found what he describes as (1) Random occurrences, (2) Observed occurrences, (3) Utmost possibilities. He used for the first part of the investigation three categories only, namely arch, loop, and whorl. The types of 101 (by mistake for 100) couplets of prints of the right forefingers of school children were taken from a large collection, the two members A and B being picked out at random and formed into a couplet. These were tabled and found to agree very well with calculated random couplets.

To study the fraternal relationship the observed occurrences were then taken from 105 fraternities and finally the greatest number of correspondences which could (without changing the total distribution of types) occur if the kinship were as close as possible were found. I now reproduce the results obtained by Galton which I take from p. 176 of his book. It will be seen that he only deals with the diagonal of his correlation table in this summary of the results.

	A and B both being		
	Arches	Loops	Whorls
Random	1·7	37·6	6·2
Observed	5·0	42·0	10·0
Utmost possible ...	10·0	61·0	25·0

We note that in all three cases the observed exceed the random but that this excess is not very marked, particularly in loops. Galton says that many other cases of this description were calculated all yielding the same general result. He

* He made an appeal for material in the pages of *Biometrika*, Vol. II. p. 356, 1903, and the letters accompanying the data he received show how active he was in the matter of collecting during 1903 and 1904.

then sub-divided the patterns under 53 heads and an experiment was made with the fore, middle and ring fingers of 150 paternal couples by Mr F. Howard Collins. The resulting table shows the same thing but the multiplicity of sub-divisions, especially when the number of pairs used is comparatively small, makes it even more difficult to assign any degree of correlation though some undoubtedly exists. Seventeen sets of twins show an even closer resemblance, two sets agreeing in all their three couplets, four agreeing in two and five in one. There are instances of partial agreement in five others.

The knowledge of the inheritance of type of finger-print was at this stage in 1903 when Galton began to collect the material which forms the chief part of the data used in this paper. The total number of persons whose finger-prints were taken was 2,300 but many of these are isolated individuals and of no use for the study of inheritance, but when these are omitted we have over 400 cases in the tables dealing with maternal inheritance, over 300 in the paternal tables and between 800 and 900 in the fraternal tables. The most complete family schedules were those collected by Mr Perrycoste from the inhabitants of Polperro, Cornwall, and in this material we have an extraordinarily interesting series. In some cases we have the finger-print of some old man and his wife, the finger-prints of all their living children, of the husbands and wives of these children and of many of the old folks' grandchildren. The people of Polperro evidently in 1900 had not started restricting their birth-rate and we have families of eight and nine brothers and sisters with as many children again of the married ones. In Galton's later collection the finger-prints of the first fingers only were taken and the reason for this limitation is given in the schedule sent out from which I quote the following.

"It is proposed to confine the collection to prints of the two *forefingers* of many persons, to *rolled* impressions of them, and to ask for *four* prints of each.

The two forefingers are selected because their patterns are more varied than the rest in respect to 'radial' direction, and not less varied in other respects. Also because rolled impressions can be obtained most easily from them, requiring as they do the extension of the finger in use, while the others are tightly closed to be out of the way. Lastly, the degree of closeness of correlation between the two forefingers has been found to be very suitable to the inquiry."

Unluckily experience has shown us that it is more satisfactory to work with the prints of all the fingers of both hands, so I also considered the larger and earlier general finger-print collection made by Galton and I have found therein about 350 individuals who had a brother or a sister recorded; in this case all their finger-prints had been taken. It was on this collection that most of the published work done by Sir Francis on finger-prints was based, but I am not clear whether the special material he used for the early consideration of inheritance was taken from this data or not. If it were then, I think, only a selection of the available pairs of brothers was extracted unless the general collection was added to in more recent years after the publication of his book in 1892.

I have therefore had at my disposal two series of data (1) the earlier one, in which finger-prints of every finger were taken, consisting of about 370 pairs of brothers and sisters where only the fraternal relationship can be discussed, (2) the later series in which only the first finger was considered but from which we can find, not only the fraternal relationships but also the paternal and maternal and to a certain extent the relationship between cousins and the relationships between uncles and aunts and their nephews and nieces.

(2) *Classification of Finger-Prints.*

Nearly 800 of his later collection were classified by Galton and he attempted new methods of counting the ridges in an effort to obtain a numerical scale for all types of finger-prints, not only for loops. He failed, however, to find a quantitative scale that satisfied him and he put the material completely aside some years before his death. In the classification of types Galton used great detail and an examination of the detail is most suggestive. One of the questions that arises in a study of finger-prints is, as we have already indicated, whether the types of finger-print show a continuous variation; we are accustomed to think in the first place of three distinct patterns, arch, loop and whorl; but a minute examination of finger-prints at once reveals many intermediate steps and Galton in a very early stage of his work recognized many more groups than these and laid great stress on what he called "transitional cases." In 1892 he sub-divided patterns under 53 heads and the reproduction of these patterns forms a most interesting series. Of these transitional cases the most common occur among loops and all stages between loops and whorls can be found. Galton in his later work uses four types of loops (*a*) plain loops, (*b*) eyed loops "y," (*c*) invaded loops "v" and (*d*) hooked loops "k." Of these "y" and "v" are the more common and "y" is obviously the first stage in a whorl. Central pocket loops as Henry calls them seem to be another stage, but it is quite easy to find several finger-prints that might be considered either "y" loops or central pocket loops. Francis Galton wrote in his *Finger-Print Directories*, 1895, p. 100: "The transitional cases between loops and whorls are largely due to the presence in loops of a strongly marked 'y,' doubt arising whether its outline does or does not make a complete circuit." Galton used the word "compound" to cover these transitional cases, but the word has been superseded by "composite" which includes those cases compounded of loop and whorl but where I think whorl is uppermost, for composites do not include what I might describe as "not plain" loops.

By a study of the reproductions between pp. 74 and 75 in Galton's *Finger-Prints* we can see how arches pass into loops through "forked" and "tented" arches and how "eyed" loops (Galton used the term "eyeletted" in 1892) become central pocket loops and central pocket loops become whorls. There is no difficulty in making 53 or more groups of finger-prints; the difficulty arises when we have to agree where arches end and loops begin and where loops end and whorls begin. Galton found this a difficulty when Mr Howard Collins and he were working together. He writes: "living at a distance apart, it was not easy at the time

they were made, to bring our respective interpretations of transitional and of some of the other invaded loops, into strict accordance, so I prefer to keep his work, in which I have perfect confidence, independent from my own."

On the basis of my own experience I should prefer to arrange finger-prints in five groups and in the following order, arch, plain loop, "not plain" loop, composite, whorl, and if I had to combine two of these groups I should put "not plain" loop with composite but I am not yet in a position to prove that this is the correct order and the difficulty of using such an order is that plain loops still remain too large a group and "not plain" loops and composites are both very small. I therefore decided to follow Dr Waite* and divide the loops in my table according to the number of ridges, i.e. into small and large loops. The method used for working out all the tables is mean square contingency, and in applying this method we must use the class index correction for grouping. This correction assumes that the categories under consideration are approximately normal and continuous and depends upon a knowledge of the "natural order." As I said before I have no doubt that the types of finger-print form a continuous variation, and that with a more extensive study of more ample data we should be able to determine this "natural order." I do not consider that we have yet found a wholly satisfactory one, but as we shall see presently the corrective factor for grouping in this data seems to depend rather more on the number in each group than on the exact order of the grouping and that we can apply this factor without a knowledge of the absolutely correct "natural order." This question of a natural order of finger-print type attracted Francis Galton's attention and on p. 178 of his *Finger-Prints* he writes:

"It would be essential in exact working that the mutual relations of the patterns should be taken into account; for example, suppose an arch to be found on the forefinger of one brother and a nascent loop on that of the other; then, as these patterns are evidently related, their concurrence ought to be interpreted as showing some degree of resemblance."

Thus the discovery of a "natural order" is not merely essential to the completeness of mathematical reduction; it arises at once when we begin to classify our characters from the standpoint of inheritance.

(3) *The provisional Natural Order of the present paper.*

We have next to investigate how far a natural order has been established if we divide loops into small and large loops† and have as our other groups, arches, composites and whorls. Certain types of patterns are associated with small and large loops respectively. Thus small loops are more nearly associated with arches and large loops with whorls. "Not plain" loops occur much more frequently in

* *Biometrika*, Vol. x. p. 419.

† Dr Waite defines a small loop as one having twelve or fewer ridges, a large loop as one with more than twelve ridges.

large loops than in small. My tables will illustrate these points. I have worked on Dr Waite's material* and have separated "not plain" loops from loops.

If we consider the number of *small* loops in the fingers of the right hand when arches occur on 0, 1, 2, 3 and 4 fingers respectively we get the following:

If 0 arches on right hand on remaining 5 fingers we have 27·7 °/₀ small loops
1 " " " 4 " " 52·4 °/₀ "
2 " " " 3 " " 59·5 °/₀ "
3 " " " 2 " " 66·7 °/₀ "
4 " " " 1 " " 93·8 °/₀ "

If a right hand has four fingers with arches, then in 15 out of 16 cases the remaining finger has a small loop, i.e. 93·8 °/₀ of small loops, while if there are 0 arches on the hand the percentage of loops on the five fingers is 27·7 showing clearly a correlation between arches and small loops. If we consider whorls in connection with large loops we obtain the following:

If 0 whorls on right hand on remaining 5 fingers we have 27·7 °/₀ large loops
1 " " " 4 " " 37·6 °/₀ "
2 " " " 3 " " 43·4 °/₀ "
3 " " " 2 " " 49·1 °/₀ "
4 " " " 1 " " 43·1 °/₀ "

Here, the greater the number of fingers with whorls the greater the percentage of large loops on the remaining fingers. I have already noted that "not plain" (eyed, invaded and hooked) loops occur more frequently in large loops than in small. If we consider the later series collected by Francis Galton we find the following on the first finger of the left hand:

Number of ridges	Males				Females			
	Plain Loops	Not Plain Loops	Totals	Percent. Not Plain	Plain Loops	Not Plain Loops	Totals	Percent. Not Plain
1, 2, 3, 4 ...	112	10	122	8·2	185	10	195	5·1
5, 6, 7, 8 ...	113	13	126	10·3	105	18	123	14·6
9, 10, 11, 12	89	18	107	16·8	114	22	136	16·2
13, 14, 15, 16	68	19	87	21·8	87	22	109	20·2
17, 18, 19, 20	22	9	31	29·0	29	11	40	27·5
21 and over	4	4	8	50·0	5	5	10	50·0

Just as whorls are associated with large rather than small loops so are "not plain" loops, which is what we should expect if the theory that "not plain" loops are a stage on the way from loops to whorls be a true one.

* I should like here to record my indebtedness to Dr Waite for allowing me to use material that he has classified and tabled for my experiments in this direction; its use has saved me much labour and I am most grateful to him.

Dr Waite correlated* the number of each type of finger-print on the whole hand with every other type and from his diagram it is clear that arches are positively correlated with small loops and negatively with every other type so that we must place arch next to loop. Large loops are positively associated with whorls and composites and negatively with small loops and arches. Composites and whorls have a positive association. Composites seem to be more nearly associated with middle sized loops than with large ones. In fact what seems to interfere with obtaining a true natural order is the arbitrary division at 12 ridges of large and small loops; the sign in some of these tables is difficult to ascertain, a difficulty that never occurs if we deal with loops with no sub-division into small and large, or with arches or whorls. As I said before, arch, loop, "not plain" loop, composite, whorl is the order I favour, but we should want larger numbers of finger-prints than are available now and even then "not plain" loops would be too small a group comparatively for satisfactory statistical treatment. Possibly in the end we shall have to combine "not plain" loops and composites, they certainly come very close together in any natural order. In the meantime there seem to be two possible practical orders (1) Arch, Small Loop, Large Loop, Composite, Whorl, and (2) Arch, Small Loop, Composite, Large Loop, Whorl. I worked out the corrective factors (class index correlations) for both orders and was pleased to discover that they were almost identical, and that the value of the factor in this data certainly depended on the number in each group rather than on the actual order. In six cases which I tried I obtained the following results:

	Arch, S. L., L. L., C., W.	Arch, S. L., C., L. L., W.
Son's Right Forefinger	·934	·934
Father's Left Forefinger	·936	·936
Mother's Left Forefinger	·941	·941
Daughter's Left Forefinger	·934	·934
Daughter's Right Forefinger	·944	·942
Mother's Right Forefinger	·943	·943

Thus in only one case is there any difference in the third decimal place so that if we are prepared to accept type of finger-print as a continuous character we can apply the corrective factor for grouping even though we have not established an entirely satisfactory natural order. I have chosen the first of the two orders to use in every case, arch, small loop, large loop, composite, whorl. In the original tables I kept central pocket loops in a class by themselves, and throughout I worked all contingency in two ways, (1) grouping C. P. L. with whorls, and (2) grouping C. P. L with composites, but the results are not significantly different

* Dr Waite in his paper on the association of finger-prints in the same individual has endeavoured to form a scale of relationship on the basis of association as measured by contingency in the prints of the several fingers. I think he is on the right path to deduce such a scale, but I believe that it cannot be done by merely comparing the numerical magnitude of the associations. It is true that contingency has no sign, but if we are to form a scale of next relationship, we must consider sign, a positive and negative association of the same magnitude being not of the same significance in a scalar relationship. I have therefore felt compelled to give signs to Dr Waite's contingency coefficients based on other considerations than those of contingency, namely by actual examination of the contingency tables themselves. This has compelled me to modify his scale of relationship.

so I have decided to give here the result for the first grouping only, which is the one that I prefer on the whole as it makes composites a rather more satisfactory group in size. I have used mean square contingency throughout and considered the correction for the number of cells only when any doubt arose as to whether ϕ^2 was significant or not but have not subtracted this correction from ϕ^2 before finding C_2*.

(4) Before I applied biometric methods to the problem of finger-prints some endeavour was made to consider the inheritance of types from the Mendelian standpoint. The difficulties attending any attempt to consider finger-prints from this point of view consist not only in the existence of transitional forms, but from the fact that there may be inheritance with change of finger. It would be exceedingly difficult to collect cases only of both parents having all their finger-prints unique in type, for example a father with 10 whorls and a mother with 10 whorls, or a father with 10 whorls and a mother with 10 arches. Yet something of this kind seems needful, if we admit transference of type from one finger in parent to another in child. Galton's later data at any rate will not help us in this matter as we have only the prints for two fingers provided. If we assume inheritance is to the *same* finger we can obtain from Galton's data the following tables for interpretation on Mendelian lines:

Inheritance of Finger-Print Type on Left Forefinger.

Father's Type.

Child's Type. / Mother's Type	Arch				Loop				Composite				Whorl				Totals
	A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	
Arch ...	1	19	2	1	18	14	6	7	1	2	—	—	1	4	—	—	76
Loop ...	1	30	5	8	35	141	24	47	7	15	6	5	6	55	2	7	394
Composite	—	1	2	—	6	9	6	11	1	4	2	5	1	5	1	3	57
Whorl ...	—	6	—	4	4	28	3	27	—	1	4	8	1	18	4	11	119
Totals	2	56	9	13	63	192	39	92	9	22	12	18	9	82	7	21	646

Inheritance of Finger-Print Type on Right Forefinger.

Father's Type.

Child's Type. / Mother's Type	Arch				Loop				Composite				Whorl				Totals
	A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	
Arch ...	8	6	—	3	13	25	4	10	2	4	—	—	2	5	—	—	82
Loop ...	5	27	—	14	31	147	23	32	5	15	—	3	12	46	2	9	371
Composite	1	5	—	3	5	19	3	14	1	5	—	4	—	7	2	3	72
Whorl ...	—	2	—	5	1	31	5	28	—	3	—	10	2	27	—	13	127
Totals	14	40	—	25	50	222	35	84	8	27	—	17	16	85	4	25	652

* The "correction for cells" is the mean value of the mean square contingency when the association is zero. It forms of course the proper and adequate test of zero association, but Professor Pearson tells me that he is now extremely doubtful whether it should be subtracted from the mean square contingency when the latter is really significant.

Now if we treat our four fundamental types as distinct we have the following results of crossings:

Arch × Arch	= 9 Arches +	6 Loops +	1 Composite +	0 Whorls,
Arch × Loop	= 56 Arches +	123 Loops +	17 Composites +	13 Whorls,
Arch × Composite	= 5 Arches +	17 Loops +	4 Composites +	0 Whorls,
Arch × Whorl	= 7 Arches +	40 Loops +	4 Composites +	12 Whorls,
Loop × Loop	= 39 Arches +	288 Loops +	28 Composites +	59 Whorls,
Loop × Composite	= 16 Arches +	77 Loops +	18 Composites +	12 Whorls,
Loop × Whorl	= 26 Arches +	180 Loops +	37 Composites +	100 Whorls,
Composite × Composite	= 0 Arches +	6 Loops +	2 Composites +	4 Whorls,
Composite × Whorl	= 0 Arches +	12 Loops +	12 Composites +	22 Whorls,
Whorl × Whorl	= 0 Arches +	16 Loops +	6 Composites +	24 Whorls.

These are the equations which the Mendelian has to interpret. Certain things are at once obvious:

(i) That both Arch × Arch and Arch × Composite appear to give no whorls.

(ii) That both Whorl × Whorl and Whorl × Composite appear to give no arches.

(iii) That Arch × Loop, Whorl × Loop, Composite × Loop and therefore not unnaturally Loop × Loop can reproduce all four types.

(iv) That the Composite is so relatively rare that it is difficult to lay stress on the numbers obtained for it. But Composite × Composite suggests on meagre evidence that this mating cannot produce arches.

(v) That Arch × Whorl can reproduce all types.

These results suggest that the loop is of a much more heterozygous character than either arch or whorl. If we throw composites in with whorls, the Composite × Whorl by no means gives the same result as Whorl × Whorl; and if we throw composite in with loop we do not find the equations for Composite × Loop, Arch or Whorl in satisfactory agreement with those for Loop × Loop, Arch or Whorl. The evidence, meagre as it is, seems to indicate a real genetic differentiation between Composite and Loop.

Still if we venture to put Composite with Loop, we obtain a much simplified series of equations, which admit of being expressed as follows in *round* numbers:

				Ratios
(i) Arch × Whorl	= 6 Arches +	42 Loops +	12 Whorls,	(1 : 7 : 2)
(ii) Arch × Arch	= 30 Arches +	30 Loops +	0 Whorls,	(1 : 1 : 0)
(iii) Arch × Loop	= 16 Arches +	40 Loops +	4 Whorls,	(4 : 10 : 1)
(iv) Loop × Loop	= 6 Arches +	45 Loops +	9 Whorls,	(2 : 15 : 3)
(v) Loop × Whorl	= 4 Arches +	38 Loops +	18 Whorls,	(1 : 10 : 4)
(vi) Whorl × Whorl	= 0 Arches +	30 Loops +	30 Whorls.	(0 : 1 : 1)

The symmetry of (ii) and (vi) and again of (iii) and (v) are possibly suggestive; while a comparison of (i) and (iv) seems to indicate that the whorl is to some degree dominant over the arch, though both whorl and arch must be heterozygous in their nature. We have not succeeded in putting any satisfactory interpretation on the above equations, although we have made a variety of attempts. It is possible that some one better versed than we are in handling Mendelian ratios will be able to throw more light on the matter. But we feel that the above equations are in themselves forced because:

(*a*) there is no real justification for putting the Composite with the Loop,

(*b*) there are no really clear-cut categories, but a great variety of types with transitional forms, and

(*c*) there exists good evidence to show that the type is not necessarily inherited from one finger of the parent to the same finger of the child.

Failing a Mendelian interpretation, which appeared to us at all satisfactory, we have discussed the inheritance in this paper solely from the biometric standpoint.

(5) *Parental Correlation in Finger-Print Types.*

We will first consider the later and larger series of data collected by Francis Galton in which prints were taken of the first finger only and deal with the parental tables.

TABLE I.

	Right first finger		Left first finger	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Father and Son ...	·291	·333	·379	·434
Father and Daughter	·321	·363	·236	·273
Mother and Son ...	·369	·423	·366	·416
Mother and Daughter	·366	·411	·341	·386
Mean Parental value	·337	·383	·334	·382

I have not worked out the true probable error of these contingency values. If we use the probable error of r it is of the order of ·03; contingency would be higher but as all the values of the coefficients are obviously significant, it seemed needless to undertake the labour of finding the true probable error. The mean parental value for both fingers is ·38 when corrected for grouping, which is a lower value than that generally found for the relationship between parent and child. The coefficient between the left finger of father and daughter is distinctly lower than any other of the coefficients in the table, and this difference exists whatever grouping I use. I was at first very interested to find this smaller value for the father and daughter as Francis Galton in his study of finger-prints

thought he traced some preponderance of maternal influence in the transmission of finger patterns but the value found for left first finger of father and son upsets this theory, and later work by Mr Howard Collins casts some doubt on the earlier results he had obtained which Galton quoted*. At the same time I thought it would be of interest to compare the finger-print of the mother with all her children irrespective of sex and of the father with all his children.

The advisability of combining the sexes is discussed later in this paper; it is certainly legitimate for the first finger of the left hand but I am doubtful whether it is for the first finger of the right hand. I give the values I obtained in Table II.

TABLE II.

	Father		Mother	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Right first ...	·261	·297	·357	·405
Left first	·289	·332	·338	·383

In both cases the value is higher for the mother than for the father; the probable error of the maternal coefficients is of the order ·020 and of the paternal ·023 and of the difference ·03, and from this we might infer that in the right first finger the difference was significant, but since in the right first finger I am very doubtful if I ought to have combined the sexes I think that we cannot assume on this data a preponderance of maternal influence in the inheritance of type of finger-print. A comparison of the results with those found for stature, span and forearm† shows a distinctly lower value for inheritance of finger-prints.

TABLE III.

Character	Father and		Mother and	
	Son	Daughter	Son	Daughter
Stature	·514 ± ·015	·510 ± ·013	·494 ± ·016	·507 ± ·014
Span	·454 ± ·016	·454 ± ·014	·457 ± ·016	·452 ± ·015
Forearm	·421 ± ·017	·422 ± ·015	·406 ± ·017	·421 ± ·015
Mean ...	·463	·462	·452	·460

* *Finger-Prints*, p. 190.

† "On the Laws of Inheritance in Man," by K. Pearson and A. Lee, *Biometrika*, Vol. II. pp. 357 et seq. 1903.

The only physical character in parent and child with which the inheritance of type of finger seems to agree is that of general health where the following coefficients were found.

TABLE IV.

	Corrected for grouping only
Father and Son	·438
Father and Daughter ...	·314
Mother and Son	·401
Mother and Daughter ...	·393
Mean	·387

When no correction for the number of cells is applied, the parental inheritance of general health and of type of finger appear practically the same, but in the case of general health there is a reason for the low value found which does not apply in the other case, and the resemblance is apparent rather than real. The larger number of deaths among delicate children is probably sufficient to account for the lower parental value but the whole question is discussed in detail in *Biometrika**.

(6) *Fraternal resemblance.*

In this case we have two sets of data (1) the long series for the first finger only and (2) the series for the whole hand. In the long series which we will consider first, there are 840 pairs of brothers, 886 pairs of sisters and 869 pairs of brothers and sisters.

TABLE V.

	Right first finger		Left first finger	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Brother-Brother ...	·369	·424	·335	·385
Sister-Sister ...	·316	·356	·305	·348
Brother-Sister ...	·336	·382	·340	·387
Mean	·340	·387	·327	·374

The probable error, if we use the formula for the probable error of r, is of the order ·02 and the values are very similar to those found for the paternal and maternal correlation. The correlation between sisters is the lowest for both right

* "On the Hereditary Character of General Health," by Karl Pearson and Ethel M. Elderton, *Biometrika*, Vol. IX. pp. 320 et seq.

and left finger but is certainly not significantly different for the left hand; it might be just significant if we compare sister-sister and brother-brother results for the right first finger; but I think not, considering that the probable error of a contingency coefficient is decidedly higher than the probable error of r. Before comparing these results with those found for other characters we will consider the results from the series in which the whole hand was taken. There are roughly 380 males and 230 females. I decided to put males and females together if possible and work with sibships. I had therefore to see first whether the distribution of the types of finger-prints differed sensibly, as, if it did, the sexes could not be combined. I used the method described by Professor Pearson in *Biometrika** and found the following values for χ^2 and P for the different fingers. There were five groups and I worked out χ^2 and P in the double classification, including Central Pocket Loops first with composites and secondly with whorls.

TABLE VI.

	Including C. P. L. with Composites		Including C. P. L. with Whorls	
	χ^2	P	χ^2	P
R_1	9·21	·06	9·93	·04
R_2	10·87	·03	12·08	·02
R_3	4·04	·40	4·72	·32
R_4	1·47	·83	2·34	·68
R_5	5·91	·21	4·59	·34
L_1	3·89	·42	3·87	·43
L_2	·97	·91	1·26	·87
L_3	4·03	·40	5·50	·24
L_4	3·70	·45	4·02	·41
L_5	6·57	·16	5·33	·26

It is clear from this table that, whichever grouping is used, there is no reason why the sexes should not be put together except for the thumb and first finger of the right hand. For the right thumb the odds are roughly 17 to 1 or 24 to 1 according to the grouping used against the males and females being random samples of the same population and for the right first finger the odds are roughly 59 to 1 or 33 to 1 against. This sex difference in the first finger of the right hand is confirmed by the series in which only the first finger was taken. I find from that series that for the first left finger when C. P. L.'s are grouped with composites $\chi^2 = 2{\cdot}96$ and $P = {\cdot}56$ but for the right first finger $\chi^2 = 9{\cdot}86$ and $P = {\cdot}04$ and the odds are about 23 to 1 against males and females being random samples of the same population in *these two series*. That two independent series should both support the suggestion that there is a sexual difference between the finger-print type of the right forefinger (possibly also of the right thumb) is very noteworthy, and the possibility of it should be borne in mind and retested on

* "On the Probability that Two Independent Distributions of Frequency are really samples from the same Population," by Karl Pearson, F.R.S., *Biometrika*, Vol. VIII. pp. 250—254, July 1911.

larger numbers. The biological origin of such a difference in one (or two) out of the ten fingers is not obvious, but it is the sort of clue which should not be allowed to drop. At the same time I am doubtful if it will be confirmed eventually that there is a sex difference. In both series the higher value found for χ^2 is due mainly to a disproportion in the number of arches between males and females, but in the shorter and earlier series the males have a larger proportion of arches than the females and in the later and longer series the females have a larger proportion than the males! A recount, after an interval of three or four years, confirmed the original figures, and I think the discrepancy may be due to the fact that in both series we are dealing with family finger-prints and a predominance of a certain type of finger-print in a large family of males or females may be the source of the divergency. For the right thumb there is no other series with which to compare it, but I should certainly lay no stress on the sex difference in that case; again arches are mainly responsible for the high value found for χ^2, and as arches are very rare on the thumb the percentages of arches are based on 7 males and 11 females and with such small numbers are liable to a large error. This sex difference may or may not exist; on the available data I think it would be unwise to be dogmatic on the point. Whether or not there is a sex difference there is a difference in the right thumb and right first finger in this data and under the circumstances I was reluctant to put males and females together, but the number of pairs of sisters is so small that I decided not to keep them separate. I sought justification for this step in the following way. I worked out the correlation coefficients between each finger and every other finger for (*a*) males alone, (*b*) males and females together and found that in the tables involving the right thumb and the right first finger the differences between the coefficients found were not significant and I trust that the results for inheritance in the case of the right thumb and right first finger will not be upset by dealing with sibships and neglecting the difference in sex. Table VII gives the differences in the correlation coefficients and the probable error of the difference.

Again I have used the p.e. of r which is an underestimate of the probable error of a contingency table and if the differences are not significant when this

TABLE VII.

	Right Thumb	Right First Finger
Left Thumb ...	·001 ± ·021	·035 ± ·036
First ...	·034 ± ·036	·009 ± ·026
Second ...	·069 ± ·037	·031 ± ·027
Third ...	·016 ± ·035	·028 ± ·033
Fourth ...	·053 ± ·033	·001 ± ·036
Right Thumb ...	—	·049 ± ·036
First ...	·049 ± ·036	—
Second ...	·043 ± ·036	·057 ± ·028
Third ...	·037 ± ·034	·009 ± ·033
Fourth ...	·056 ± ·033	·026 ± ·033

probable error is used we can be quite sure they will not be significant if we used the true probable error.

I will now give the inheritance coefficients for each finger for the sibships; the results are summarized in the following table.

TABLE VIII.

Sibships	Right hand		Left hand	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Thumb	·320	·379	·425	·477
First finger ...	·340	·389	·361	·411
Second finger ...	·331	·384	·443	·502
Third finger ...	·402	·477	·358	·411
Fourth finger ...	·368	·439	·366	·440
Mean ...	·352	·413	·391	·449

If we compare the values found for the first finger with the mean values found for the first finger in the longer series we find they are in close agreement, but the agreement is less close in the left hand ·374 and ·411 being the values; the difference between them is, however, not significant. It will be interesting to see whether there is any significant difference in inheritance in the different fingers. Again I have not worked out the p.e. of the contingency but taking the p.e. of r as a rough measure we find that all the probable errors in this table lie between ·020 and ·023 and that the probable error of the difference between any two coefficients or any one coefficient and the mean is about ·03. As this probable error is an underestimate of the true probable error it will be wiser to take three times the probable error and consider that unless differences in the coefficients exceed ·09 they are unlikely to be significant. No coefficient in either hand differs from its mean by more than ·064 when the correction for grouping has been made, and if we take all ten coefficients for both hands we find that the mean is ·431, and we see that no coefficient exceeds this mean by more than ·071 which is certainly not significant. The greatest difference in the coefficients found are in the right thumb and in the second left finger where the difference is ·123. A comparison of all the values for fraternal correlation shows that those for finger-prints are lower than we should expect. There is a table of fraternal resemblances on p. 25 in *Nature and Nurture, The Problem of the Future* by Karl Pearson* and the lowest fraternal coefficient is that of brother and sister for the Cephalic Index (·43). In certain fingers when the whole hand is considered one or two values approach to what we should expect, as for instance the third right finger, the second left finger and the left thumb, but on the whole the values are lower than we now expect to find when we consider fraternal resemblance.

* *Eugenics Laboratory Lecture Series*, No. VI. 1910. Cambridge University Press.

(7) *Assortative Mating.*

In this connection it would be as well to point out that in the data we have there is no tendency for like to marry like and that the assortative mating coefficient for type of finger-print is nil. There are only 177 cases where the finger-prints of husband and wife are given, and the ϕ^2 becomes ·0615 and ·0536 for left and right first finger respectively and the correction for the number of cells is ·0904 for both coefficients; accordingly the ϕ^2 is insignificant.

TABLE IX.

	First Right Finger						First Left Finger					
	Husband						Husband					
Wife	Arch	S. Loop	L. Loop	Comp.	Whorl	Totals	Arch	S. Loop	L. Loop	Comp.	Whorl	Totals
Arch	1 1·82	8 9·49	6 5·07	1 1·17	7 5·46	23	1 2·82	14 10·88	3 3·53	3 1·84	4 5·93	25
S. Loop	6 5·69	30 29·39	16 15·86	3 3·66	17 17·08	72	9 6·89	23 26·54	9 8·62	3 4·48	17 14·47	61
L. Loop	4 2·77	12 14·44	8 7·71	1 1·78	10 8·31	35	3 3·28	13 12·62	4 4·10	2 2·13	7 6·88	29
Comp.	— ·47	4 2·47	2 1·32	— ·31	— 1·42	6	1 1·69	7 6·53	1 2·12	3 1·10	3 3·56	15
Whorls	3 3·24	19 16·91	7 9·03	4 2·08	8 9·73	41	6 5·31	20 20·45	8 6·64	2 3·45	11 11·15	47
Totals	14	73	39	9	42	177	20	77	25	13	42	177

The black figures give the independent probability and it can be seen quite easily that the differences between the independent probability and the actual figures are small and the signs indiscriminate; there are no well marked positive differences along the diagonal and it is fairly obvious that there is no evidence of assortative mating in finger-prints. It will be clear therefore that assortative mating had no influence on our results.

(8) *Probable Explanation of the low values of the coefficients of inheritance in the case of Finger-Prints.*

Galton's material for parental heredity limits us to the consideration of what happens in the case of the forefingers of the two hands. But this limitation compels us to discuss only the case of inheritance from like finger to like finger. Now let us suppose that the finger-prints of both parents were entirely whorls, but that the offspring had composites on the two forefingers and the remaining fingers all whorls. There would clearly here be a marked inheritance of very anomalous distribution, but an investigation which confined itself to the forefingers would fail or largely fail to measure its intensity. If we deal solely with like fingers, we overlook

the possibility that the inheritance of the type may not be from like finger to like finger alone, but that the type may be transferred to some other finger of the offspring. And it is fairly easy to demonstrate that there is an association not only between the types of like fingers in parent and child but also between the types of unlike fingers. In other words if a very rare type, say, occurs on the forefinger of the parent, it may more frequently appear on the same finger of the offspring, but should it appear on another finger of the offspring, we cannot pass it by and assume that its appearance is independent of heredity. Clearly we should not do so if polydactyly occurred in the right hand of a parent, but in the left hand or even either foot of the offspring! We must therefore suppose a correlation ρ_{ss} between like fingers in parent and offspring and another correlation $\rho_{ss'}$ between unlike fingers. Our treatment thus far would only be satisfactory provided $\rho_{ss'} = 0$. If this were true descent would be from like finger to like finger. If $\rho_{ss'} = \rho_{ss}$ descent would be indifferent as to finger. *A priori* we should anticipate that $\rho_{ss'}$ would not be zero but $< \rho_{ss}$, or there would be some bias in favour of the type being transferred to like finger. In collecting a large mass of material by voluntary help it was clearly needful to reduce to the utmost the *desiderata*. But in the present case Galton's restriction of his data to the simplicity of forefinger prints has sadly limited the possibility of our determining the real strength of finger-print inheritance. Like all preliminary inquiries, however, it has sufficed to indicate what we need at the next stage, namely complete sets of family finger-prints, and it is to the heavier task of such a collection that the Galton Laboratory is now addressing itself. The theory of the general inheritance of finger-prints may be attacked in the following manner.

Let u be a variable which depends on 10 other variables $x_1, x_2 \ldots x_{10}$ (i.e. the finger-print types of the ten fingers) and u' be the like variable in a second individual. Now if we suppose the variability in type of each finger to be the same—which is only approximately true—and each finger to contribute equally to u, we can take as appropriate values:

$$u = \tfrac{1}{10}(x_1 + x_2 + \ldots + x_{10}),$$
$$u' = \tfrac{1}{10}(x_1' + x_2' + \ldots + x'_{10}).$$

We will further suppose all our variates measured from their means.

Then $r_{uu'}$ will be the correlation between the characters u and u' in the two individuals.

Further if σ denote a standard deviation.

$$\sigma_u^2 = \tfrac{1}{100}\{\sigma^2_{x_1} + \sigma^2_{x_2} + \ldots + \sigma^2_{x_{10}} + 2S\sigma_{xs}\sigma_{xs'}r_{ss'}\}$$
$$= \frac{\sigma_x^2}{10}\left\{1 + \frac{2S(r_{ss'})}{10}\right\}.$$

If $\bar{r}_{ss'}$ be the mean value of $r_{ss'} = \frac{1}{45}S(r_{ss'})$, we have

$$\sigma_u^2 = \tfrac{1}{10}\sigma_x^2(1 + 9\bar{r}_{ss'}) = \sigma_{u'}^2,$$

supposing u and u' are individuals in a stable population.

Again if N be the total number of pairs

$$\frac{1}{N}S(uu') = \frac{\sigma_x^2}{100}\left\{S(\rho_{ss}) + S(\rho_{ss'})\right\},$$

where ρ_{ss} is the correlation of x_s and x'_s and $\rho_{ss'}$ of x_s and $x'_{s'}$. Accordingly if $\bar{\rho}_{ss} = \frac{1}{10}(\rho_{11} + \rho_{22} + \ldots + \rho_{10\,10})$, and $\bar{\rho}_{ss'} = \frac{1}{90}S(\rho_{ss'})$, i.e. are the mean correlations, we have

$$\frac{1}{N}S(uu') = \tfrac{1}{10}\sigma_x^2(\bar{\rho}_{ss} + 9\bar{\rho}_{ss'}).$$

Thus we obtain finally:

$$r_{uu'} = \frac{\bar{\rho}_{ss} + 9\bar{\rho}_{ss'}}{1 + 9\bar{r}_{ss'}}.$$

Here $\bar{r}_{ss'}$ is a mean of organic correlations in the same individual, and $\bar{\rho}_{ss}$ and $\bar{\rho}_{ss'}$ are mean direct and cross coefficients of heredity.

If inheritance be unassociated with likeness of finger, we have $\bar{\rho}_{ss'} = \bar{\rho}_{ss}$ and accordingly

$$r_{uu'} = 10\bar{\rho}_{ss}/(1 + 9\bar{r}_{ss'}).$$

If inheritance is only concerned with like fingers $\bar{\rho}_{ss'} = 0$ and

$$r_{uu'} = \bar{\rho}_{ss}/(1 + 9\bar{r}_{ss'}).$$

If there were equal heredity for any characters or function of characters then we must have $r_{uu'} = \bar{\rho}_{ss}$, which leads directly to

$$\bar{\rho}_{ss'} = \bar{\rho}_{ss} \times \bar{r}_{ss'}$$

the relation suggested by Pearson for cross-heredity, i.e. the correlation between two *different* characters in parent and offspring*. It seems, however, that there is extremely likely to be a closer relationship than that of cross-heredity between finger-prints on different fingers, and the only means of testing this point is to measure directly $\bar{\rho}_{ss}$ and $\bar{\rho}_{ss'}$.

Unfortunately Galton's later and longer series will not here help us. We are compelled to turn to his earlier series, not gathered for hereditary purposes, but containing—so to speak—accidentally some 600 pairs of siblings with all ten finger-prints given in most of our individual records. I decided to find $\bar{r}_{ss'}$ from that data; then assuming that inheritance of type was indifferent as to finger we could obtain limits between which the true correlation coefficient of type of finger-print might be said to lie. As before I included Central Pocket Loops in Composites. I combined males and females and used the correction for grouping. The results obtained are given in Table X. There are about 600 cases. The mean (taken from four figures) was found to be ·6076. We can now proceed to apply this corrective factor to the coefficients we have found using the coefficient corrected for grouping. We are assuming for the moment that $\bar{\rho}_{11} = \bar{\rho}_{12}$, then the coefficients already obtained must be multiplied by 1·54598. The results are given in Tables XI and XII. Now all

* *Phil. Trans.* Vol. 197, p. 290, *R. S. Proc.* Vol. 62, p. 410, and *Biometrika*, Vol. II. p. 385.

TABLE X. *Galton's Earlier Data. Correlation of Different Finger Types in the same Individual.*

Males and Females.

	L. Thumb	L_1	L_2	L_3	L_4	R. Thumb	R_1	R_2	R_3	R_4
L. Thumb	1	—	—	—	—	—	—	—	—	—
L_1 ...	·559	1	—	—	—	—	—	—	—	—
L_2 ...	·566	·653	1	—	—	—	—	—	—	—
L_3 ...	·470	·560	·674	1	—	—	—	—	—	—
L_4 ...	·577	·532	·503	·708	1	—	—	—	—	—
R. Thumb	·825	·518	·511	·539	·565	1	—	—	—	—
R_1 ...	·510	·728	·694	·576	·547	·516	1	—	—	—
R_2 ...	·510	·686	·786	·653	·475	·514	·665	1	—	—
R_3 ...	·525	·541	·608	·772	·685	·598	·589	·619	1	—
R_4 ...	·606	·593	·577	·744	·815	·556	·593	·584	·717	1

TABLE XI.

	Right First Finger		Left First Finger	
	r	*R*	*r*	*R*
Father and Son	·333	·515	·434	·671
Father and Daughter ...	·363	·561	·273	·422
Mother and Son	·423	·654	·416	·643
Mother and Daughter ...	·411	·635	·386	·597
Mean	·383	·592	·382	·591

TABLE XII.

	Sibship		Brother-Brother		Sister-Sister		Brother-Sister	
	r	*R*	*r*	*R*	*r*	*R*	*r*	*R*
Right Thumb ...	·379	·586	—	—	—	—	—	—
First ...	·389	·601	·424	·655	·356	·550	·382	·591
Second ...	·384	·594	—	—	—	—	—	—
Third ...	·477	·737	—	—	—	—	—	—
Fourth ...	·439	·679	—	—	—	—	—	—
Mean ...	·413	·638	—	—	—	—	—	—
Left Thumb ...	·477	·737	—	—	—	—	—	—
First ...	·411	·635	·385	·595	·348	·538	·387	·598
Second ...	·502	·776	—	—	—	—	—	—
Third ...	·411	·635	—	—	—	—	—	—
Fourth ...	·440	·680	—	—	—	—	—	—
Mean ...	·449	·694	—	—	—	—	—	—

these new coefficients are distinctly higher than we should expect but depend on the assumption that inheritance of type is indifferent as to finger. Now if this be true, the correlation coefficient between any one finger of a sibling and all the other fingers in a second sibling should be the same; but if inheritance of type depends absolutely on the one finger as I have assumed by correlating each finger separately there would be no correlation between one finger and any other finger in a sibship. I decided to test this on the available data in two cases, namely for the left thumb and for the left first finger. I did not go further than this for several reasons, chiefly because the labour involved is very great, and since I am doubtful about the wisdom of combining males and females it seemed waste of time to continue till we had a larger series of data for the whole hand, and were not entirely dependent on about 360 pairs of brothers and sisters from Galton's earlier data. The cross tables were made so that each pair of siblings gave two entries. The numbers in each table averaged just over 700. The following coefficients were obtained.

TABLE XIII.

Left Thumb of first Sibling and			Left Thumb	of	second Sibling	·477
„	„	„	First	„	„	·368
„	„	„	Second	„	„	·309
„	„	„	Third	„	„	·235
„	„	„	Fourth	„	„	·314
Left Thumb of first Sibling and			Right Thumb	„	„	·410
„	„	„	First	„	„	·316
„	„	„	Second	„	„	·267
„	„	„	Third	„	„	·344
„	„	„	Fourth	„	„	·269

It will be seen at once that all these coefficients are significant but all are lower than the correlation between the same finger in siblings*; that is to say inheritance of type is partially indifferent as to finger but not wholly so, and to find the true measure of correlation between type of finger-print we should correct by using the formula $\frac{\bar{\rho}_{11} + 9\bar{\rho}_{12}}{1 + 9\bar{r}_{12}}$. Adopting this correction for the right thumb we find that $R = \frac{\cdot 477 + 2\cdot 832}{1 + 5\cdot 4684} = \cdot 512$ which is a very reasonable value for the coefficient of correlation between siblings.

I did the same thing for the left first finger, and found the results in Table XIV.

These results agree fairly well with the results previously found for the resemblance between brothers and I think it is reasonable to assume that when we can apply similar corrections to the parental tables we shall reach results corresponding equally well. In other words it is extremely probable that finger-prints are in-

* It will be noted at once that like fingers have the highest, homologous fingers the next highest correlation.

herited at the same rate as other physical characters, but the type characteristic of any finger in the parent while mostly likely to pass to the like finger in the offspring, may easily pass to the homologous finger of the other hand or indeed to any other finger whatever of the offspring.

TABLE XIV.

Left	First	Finger of	first	Sibling and Left	First	Finger of second Sibling	·411
„	„	„	„	Thumb	„	„	·368
„	„	„	„	Second	„	„	·387
„	„	„	„	Third	„	„	·357
„	„	„	„	Fourth	„	„	·345
Left	First	Finger of	first	Sibling and Right	Thumb	„	·314
„	„	„	„	First	„	„	·399
„	„	„	„	Second	„	„	·342
„	„	„	„	Third	„	„	·384
„	„	„	„	Fourth	„	„	·354

and $$R = \frac{\cdot 411 + 3 \cdot 258}{1 + 5 \cdot 4684} = \cdot 567.$$

(9) It is possible from the finger-prints in Galton's later series to obtain for the first fingers the correlation between grandparents and their grandchildren, between uncles and aunts and their nephews and nieces and between pairs of cousins. For the consideration of the grandparental relationship we have only 405 cases and to obtain even this small number we have had to combine the sexes. In the uncle and aunt tables we have rather over 500 cases in each table so we could keep the sexes separate but could not distinguish between paternal and maternal relations. We found 657 pairs of female cousins, 795 pairs of males and 1510 of male and female.

Until we have data for every finger it is not possible to deal adequately with the inheritance of type of finger-print as we cannot allow for the fact that inheritance of type of finger-print is not wholly indifferent as to finger but we give the results in Table XV; it seems especially needful to give them since they do not wholly confirm results previously reached; cousins show a decidedly lower value than we should expect.

For grandparents the values found for right and left forefingers are practically identical; the values given on p. 78 are below the true value and when we can make the necessary adjustment we shall probably find that the grandparental coefficient for the inheritance of type of finger-print is about ·3. The only grandparental data at present reduced for man are those for eye colour and the eight cases give a mean value of ·32 with which the value found for finger-prints agrees very well but both these are somewhat higher than the mean value found for other species. We are hoping soon to work out the grandparental coefficients for other characters.

In the uncle and aunt tables the coefficients found for the right forefinger are lower than those found for the left but the difference is not significant. The value

TABLE XV.

	First Left		First Right	
		Corrected for grouping		Corrected for grouping
Grandparent and Grandchild	·237	·281	·246	·285
Uncle and Nephew	·302	·342	·291	·337
Uncle and Niece	·259	·297	·211	·245
Aunt and Nephew	·266	·303	·229	·262
Aunt and Niece	·280	·323	·192	·217
Mean	—	·316	—	·265
Male Cousins	·115	·131	·210	·244
Female Cousins	·152	·174	·130	·145
Male and Female Cousins ...	·109	·124	·137	·156
Mean	—	·143	—	·182

of ·316 is higher than we should expect from other work done and this will be increased when we can allow for the inheritance between the first finger and the other fingers. If we compare the mean of the two forefingers ·29 with that found for the grandparental relationship we see that they are almost the same and we had supposed from previous work that the grandparental relationship was slightly higher. At the same time considering the number of cases and the probable error there is really nothing in contradiction with former work in these results.

We cannot however say the same when we turn to cousins. Inheritance of type of finger-print in cousins is significantly lower than in the grandparental or avuncular relationship, a result not confirmed by any other work we have done except in the less easily measured characters such as temper. Though the values of the coefficients are very low the diagonals are positive and there is no doubt about the sign which is an important point to consider when the contingency values, uncorrected for grouping, are as small as they are in this table. Further the correction for the number of cells leaves ϕ^2 positive, but we must admit that we were surprised to find such low values and it is difficult to account for them. Family schedules accompanied the finger-prints and there seems very little chance that there could have been any selection of the cousins considered. When the anthropometric department at the Galton Laboratory can be fully equipped and opened we hope to obtain more extensive finger-print data for the whole hand for collaterals and until we have that material we can only give the results we have reached on Galton's own data. We have at present no explanation to offer for the low values obtained for the resemblance in finger-prints between cousins.

APPENDIX OF DATA.

GALTON'S LATER SERIES

TABLE I.

Father's Left First Finger and Son's Left First Finger.

Father.

Son.	A.	S. L.	L. L.	C.	W.	Totals
A.	13	16	3	—	1	33
S. L.	18	62	23	8	26	137
L. L.	1	25	10	4	12	52
C.	1	11	6	7	4	29
W.	8	25	11	9	25	78
Totals	41	139	53	28	68	329

TABLE II.

Father's Right First Finger and Son's Right First Finger.

Father.

Son.	A.	S. L.	L. L.	C.	W.	Totals
A.	8	15	8	1	3	35
S. L.	19	64	22	10	25	140
L. L.	6	21	12	4	14	57
C.	3	10	4	6	9	32
W.	2	28	12	11	27	80
Totals	38	138	58	32	78	344

TABLE III.

Father's Left First Finger and Daughter's Left First Finger.

Father.

Daughter.	A.	S. L.	L. L.	C.	W.	Totals
A.	14	22	8	3	2	49
S. L.	28	68	15	16	24	151
L. L.	7	28	10	6	12	63
C.	2	12	6	5	8	33
W.	5	18	6	4	9	42
Totals	56	148	45	34	55	338

TABLE IV.

Father's Right First Finger and Daughter's Right First Finger.

Father.

Daughter.	A.	S. L.	L. L.	C.	W.	Totals
A.	7	21	5	5	5	43
S. L.	14	80	18	6	27	145
L. L.	6	25	19	6	11	67
C.	5	9	13	7	4	38
W.	7	17	15	3	15	57
Totals	39	152	70	27	62	350

TABLE V.

Mother's Left First Finger and Son's Left First Finger.

Mother.

Son.	A.	S. L.	L. L.	C.	W.	Totals
A.	16	18	3	4	3	44
S. L.	30	74	29	14	16	163
L. L.	9	18	10	8	16	61
C.	5	10	5	6	14	40
W.	7	25	17	10	37	96
Totals	67	145	64	42	86	404

TABLE VI.

Mother's Right First Finger and Son's Right First Finger.

Mother.

Son.	A.	S. L.	L. L.	C.	W.	Totals
A.	13	19	1	3	8	44
S. L.	31	72	21	6	25	155
L. L.	4	21	15	8	12	60
C.	3	17	9	4	13	46
W.	5	27	16	6	43	97
Totals	56	156	62	27	101	402

TABLE VII.

Mother's Left First Finger and Daughter's Left First Finger.

Daughter. \ Mother.	A.	S. L.	L. L.	C.	W.	Totals
A.	19	19	9	4	6	57
S. L.	21	83	45	12	35	196
L. L.	6	27	14	8	17	72
C.	6	12	5	6	12	41
W.	3	18	7	6	28	62
Totals	55	159	80	36	98	428

TABLE VIII.

Mother's Right First Finger and Daughter's Right First Finger.

Daughter. \ Mother.	A.	S. L.	L. L.	C.	W.	Totals
A.	17	22	7	5	5	56
S. L.	34	71	35	17	19	176
L. L.	7	25	19	9	17	77
C.	6	13	9	7	13	48
W.	2	16	16	5	33	72
Totals	66	147	86	43	87	429

TABLE IX.

Brothers' Left First Finger.

Second Brother. \ First Brother.	A.	S. L.	L. L.	C.	W.	Totals
A.	22	53	10	9	6	100
S. L.	53	172	40	34	52	351
L. L.	10	40	18	12	38	118
C.	9	34	12	4	19	78
W.	6	52	38	19	78	193
Totals	100	351	118	78	193	840

TABLE X.

Brothers' Right First Finger.

Second Brother. \ First Brother.	A.	S. L.	L. L.	C.	W.	Totals
A.	24	49	12	6	5	96
S. L.	49	170	39	22	60	340
L. L.	12	39	28	17	39	135
C.	6	22	17	2	29	76
W.	5	60	39	29	92	225
Totals	96	340	135	76	225	872

TABLE XI.

Sisters' Left First Finger.

Second Sister. \ First Sister.	A.	S. L.	L. L.	C.	W.	Totals
A.	30	54	10	8	10	112
S. L.	54	216	70	27	42	409
L. L.	10	70	26	14	31	151
C.	8	27	14	16	13	78
W.	10	42	31	13	40	136
Totals	112	409	151	78	136	886

TABLE XII.

Sisters' Right First Finger.

Second Sister. \ First Sister.	A.	S. L.	L. L.	C.	W.	Totals
A.	38	46	10	12	8	114
S. L.	46	174	67	43	44	374
L. L.	10	67	30	18	23	148
C.	12	43	18	10	24	107
W.	8	44	23	24	46	145
Totals	114	374	148	107	145	888

TABLE XIII.

Brother-Sister Left First Finger.

Second Sister. \ Brother.	A.	S. L.	L. L.	C.	W.	Totals
A.	39	36	7	9	14	105
S. L.	49	181	61	30	64	385
L. L.	4	45	31	14	37	131
C.	13	29	11	14	28	95
W.	9	51	25	15	53	153
Totals	114	342	135	82	196	869

TABLE XIV.

Brother-Sister Right First Finger.

Second Sister. \ Brother.	A.	S. L.	L. L.	C.	W.	Totals
A.	32	46	17	4	13	112
S. L.	43	173	45	29	64	354
L. L.	14	37	26	15	30	122
C.	15	32	24	12	32	115
W.	3	43	35	15	69	165
Totals	107	331	147	75	208	868

TABLE XV.

Siblings' Left Thumb.

Second Sibling. / First Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	4	24	14	2	3	47
S. L.	24	44	61	13	14	156
L. L.	14	61	166	26	45	312
C.	2	13	26	22	13	76
W.	3	14	45	13	58	133
Totals	47	156	312	76	133	724

TABLE XVI.

Siblings' Right Thumb.

Second Sibling. / First Sibling	A.	S. L.	L. L.	C.	W.	Totals
A.	—	8	11	3	4	26
S. L.	8	30	50	3	21	112
L. L.	11	50	170	17	82	330
C.	3	3	17	2	7	32
W.	4	21	82	7	116	230
Totals	26	112	330	32	230	730

TABLE XVII.

Siblings' First Left Finger.

Second Sibling. / First Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	38	44	18	1	8	109
S. L.	44	118	60	11	39	272
L. L.	18	60	34	9	36	157
C.	1	11	9	—	9	30
W.	8	39	36	9	66	158
Totals	109	272	157	30	158	726

TABLE XVIII.

Siblings' First Right Finger.

Second Sibling. / First Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	32	37	8	7	16	100
S. L.	37	114	50	15	44	260
L. L.	8	50	18	13	44	133
C.	7	15	13	2	8	45
W.	16	44	44	8	76	188
Totals	100	260	133	45	188	726

TABLE XIX.

Siblings' Second Left Finger.

Second Sibling. / First Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	32	41	17	2	2	94
S. L.	41	130	81	2	19	273
L. L.	17	81	142	4	25	269
C.	2	2	4	2	4	14
W.	2	19	25	4	28	78
Totals	94	273	269	14	78	728

TABLE XX.

Siblings' Second Right Finger.

Second Sibling. / First Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	8	27	13	—	5	53
S. L.	27	188	80	3	27	325
L. L.	13	80	114	3	36	246
C.	—	3	3	2	2	10
W.	5	27	36	2	14	84
Totals	53	325	246	10	84	718

TABLE XXI.

Siblings' Third Left Finger.

Second Sibling. / First Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	6	11	8	—	4	29
S. L.	11	68	56	9	29	173
L. L.	8	56	164	13	60	301
C.	—	9	13	4	12	38
W.	4	29	60	12	72	177
Totals	29	173	301	38	177	718

TABLE XXII.

Siblings' Third Right Finger.

Second Sibling. / First Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	2	8	2	—	5	17
S. L.	8	84	46	3	32	173
L. L.	2	46	118	9	79	254
C.	—	3	9	4	12	28
W.	5	32	79	12	126	254
Totals	17	173	254	28	254	726

TABLE XXIII.

Siblings' Fourth Left Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	4	15	3	1	2	25
S. L.	15	156	91	4	14	280
L. L.	3	91	192	6	21	313
C.	1	4	6	—	1	12
W.	2	14	21	1	12	50
Totals	25	280	313	12	50	680

TABLE XXIV.

Siblings' Fourth Right Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	4	12	3	—	—	19
S. L.	12	150	97	3	20	282
L. L.	3	97	170	7	35	312
C.	—	3	7	—	6	16
W.	—	20	35	6	14	75
Totals	19	282	312	16	75	704

TABLE XXV*.

Left Thumb and Left First Finger.

Left Thumb.

Left First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	23	35	19	8	8	93
S. L.	11	69	101	22	18	221
L. L.	2	10	74	18	24	128
C.	—	1	14	5	8	28
W.	—	6	61	16	62	145
Totals	36	121	269	69	120	615

TABLE XXVI.

Left Thumb and Left Second Finger.

Left Thumb.

Left Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	24	23	14	9	8	78
S. L.	8	73	98	16	29	224
L. L.	3	22	126	31	37	219
C.	—	1	6	3	7	17
W.	1	2	22	9	37	71
Totals	36	121	266	68	118	609

TABLE XXVII.

Left Thumb and Left Third Finger.

Left Thumb.

Left Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	9	7	3	—	4	23
S. L.	17	50	55	13	8	143
L. L.	6	41	126	28	42	243
C.	3	5	16	5	7	36
W.	1	17	64	23	57	162
Totals	36	120	264	69	118	607

TABLE XXVIII.

Left Thumb and Left Fourth Finger.

Left Thumb.

Left Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	5	2	—	—	19
S. L.	20	72	98	11	21	222
L. L.	4	37	137	41	70	289
C.	—	1	3	3	4	11
W.	—	2	15	11	18	46
Totals	36	117	255	66	113	587

TABLE XXIX.

Left Thumb and Right Thumb.

Left Thumb.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	15	1	1	1	—	18
S. L.	21	55	11	5	2	94
L. L.	1	49	197	12	13	272
C.	—	4	10	17	4	35
W.	—	10	51	34	101	196
Totals	37	119	270	69	120	615

TABLE XXX.

Left Thumb and Right First Finger.

Left Thumb.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	24	25	20	8	6	83
S. L.	11	65	98	18	23	215
L. L.	1	13	64	12	27	117
C.	1	5	16	6	8	36
W.	—	13	70	25	56	164
Totals	37	121	268	69	120	615

* Tables XXV—LXIX are organic correlation tables, i.e. correlation between the prints of the several fingers in the same person.

TABLE XXXI.

Left Thumb and Right Second Finger.

Right Second Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	17	15	7	5	2	46
S. L.	16	80	116	21	37	270
L. L.	3	20	116	29	48	216
C.	—	2	2	2	7	13
W.	—	3	29	12	16	60
Totals	36	120	270	69	110	605

TABLE XXXII.

Left Thumb and Right Third Finger.

Right Third Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	9	1	3	—	—	13
S. L.	17	49	50	14	10	140
L. L.	4	39	91	18	40	192
C.	1	10	15	4	3	33
W.	4	22	107	33	67	233
Totals	35	121	266	69	120	611

TABLE XXXIII.

Left Thumb and Right Fourth Finger.

Right Fourth Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	11	3	—	—	—	14
S. L.	21	79	98	13	15	226
L. L.	4	30	136	39	70	279
C.	1	2	5	1	6	15
W.	—	5	26	14	25	70
Totals	37	119	265	67	116	604

TABLE XXXIV.

Left First Finger and Left Second Finger.

Left Second Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	42	31	5	—	—	78
S. L.	38	128	34	7	17	224
L. L.	10	57	71	14	66	218
C.	1	1	5	3	6	16
W.	1	5	11	3	52	72
Totals	92	222	126	27	141	608

TABLE XXXV.

Left First Finger and Left Third Finger.

Left Third Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	16	5	2	—	—	23
S. L.	43	82	12	1	5	143
L. L.	21	88	64	11	57	241
C.	3	15	6	4	9	37
W.	10	27	42	12	71	162
Totals	93	217	126	28	142	606

TABLE XXXVI.

Left First Finger and Left Fourth Finger.

Left Fourth Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	15	4	—	—	—	19
S. L.	49	116	29	5	22	221
L. L.	26	82	73	22	86	289
C.	1	3	1	1	6	12
W.	1	7	15	—	22	45
Totals	92	212	118	28	136	586

TABLE XXXVII.

Left First Finger and Right Thumb.

Right Thumb.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	11	5	2	—	—	18
S. L.	34	51	3	—	6	94
L. L.	31	114	66	12	48	271
C.	5	10	10	1	7	33
W.	11	42	47	14	84	198
Totals	92	222	128	27	145	614

TABLE XXXVIII.

Left First Finger and Right First Finger.

Right First Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	45	32	5	—	—	82
S. L.	39	137	29	3	6	214
L. L.	5	28	49	9	26	117
C.	3	8	9	7	8	35
W.	2	17	35	9	103	166
Totals	94	222	127	28	143	614

TABLE XXXIX.

Left First Finger and Right Second Finger.

Right Second Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	35	8	2	—	—	45
S. L.	46	159	30	10	26	271
L. L.	13	47	78	9	68	215
C.	—	—	5	1	7	13
W.	—	6	13	8	43	70
Totals	94	220	128	28	144	614

TABLE XL.

Left First Finger and Right Third Finger.

Right Third Finger.	Left First Finger. A.	S. L.	L. L.	C	W.	Totals
A.	10	3	—	—	—	13
S. L.	44	78	14	1	4	141
L. L.	16	73	48	11	38	186
C.	4	17	5	—	6	32
W.	18	48	57	16	95	234
Totals	92	219	124	28	143	606

TABLE XLI.

Left First Finger and Right Fourth Finger.

Right Fourth Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	13	1	—	—	—	14
S. L.	58	121	27	9	10	225
L. L.	19	83	72	18	87	279
C.	—	4	2	—	7	13
W.	3	9	21	1	38	72
Totals	93	218	122	28	142	603

TABLE XLII.

Left Second Finger and Left Third Finger.

Left Third Finger.	Left Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	16	7	—	—	—	23
S. L.	46	79	14	—	1	140
L. L.	10	103	112	5	12	242
C.	4	11	18	6	4	43
W.	2	22	71	5	54	154
Totals	78	222	215	16	71	602

TABLE XLIII.

Left Second Finger and Left Fourth Finger.

Left Fourth Finger.	Left Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	12	6	—	—	—	18
S. L.	46	110	54	3	10	223
L. L.	16	85	134	10	41	286
C.	1	3	5	—	2	11
W.	2	8	16	2	17	45
Totals	77	212	209	15	70	583

TABLE XLIV.

Left Second Finger and Right Thumb.

Right Thumb.	Left Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	12	3	1	—	1	17
S. L.	29	50	13	1	2	95
L. L.	24	108	112	5	19	268
C.	5	11	13	2	4	35
W.	8	52	80	8	46	194
Totals	78	224	219	16	72	609

TABLE XLV.

Left Second Finger and Right First Finger.

Right First Finger.	Left Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	45	31	5	1	1	83
S. L.	27	124	56	3	2	212
L. L.	5	32	71	5	3	116
C.	—	16	16	1	4	37
W.	1	23	70	6	61	161
Totals	78	226	218	16	71	609

TABLE XLVI.

Left Second Finger and Right Second Finger.

Right Second Finger.	Left Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	31	13	1	1	—	46
S. L.	46	167	46	2	6	267
L. L.	1	40	143	2	28	214
C.	—	2	3	6	2	13
W.	—	3	25	5	36	69
Totals	78	225	218	16	72	609

TABLE XLVII.

Left Second Finger and Right Third Finger.

Right Third Finger.	Left Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	11	1	—	—	—	12
S. L.	46	77	14	1	1	139
L. L.	11	79	83	4	14	191
C.	2	15	14	1	1	33
W.	7	52	106	10	55	230
Totals	77	224	217	16	71	605

TABLE XLVIII.

Left Second Finger and Right Fourth Finger.

Right Fourth Finger.	Left Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	12	2	—	—	—	14
S. L.	48	119	50	4	4	225
L. L.	14	88	129	9	37	277
C.	—	2	8	1	4	15
W.	2	10	28	2	25	67
Totals	76	221	215	16	70	598

TABLE XLIX.

Left Third Finger and Left Fourth Finger.

Left Fourth Finger.	Left Third Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	10	6	1	1	1	19
S. L.	10	106	70	11	23	220
L. L.	3	24	157	25	80	289
C.	—	—	4	1	6	11
W.	—	2	3	2	39	46
Totals	23	138	235	40	149	585

TABLE L.

Left Third Finger and Right Thumb.

Right Thumb.	Left Third Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	7	4	3	3	—	17
S. L.	7	49	21	7	10	94
L. L.	5	67	124	14	56	266
C.	—	7	9	4	17	37
W.	4	14	87	12	75	192
Totals	23	141	244	40	158	606

TABLE LI.

Left Third Finger and Right First Finger.

Right First Finger.	Left Third Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	14	39	20	5	3	81
S. L.	9	79	90	12	22	212
L. L.	—	16	60	7	32	115
C.	—	2	12	5	15	34
W.	—	7	62	10	86	165
Totals	23	143	244	39	158	607

TABLE LII.

Left Third Finger and Right Second Finger.

Right Second Finger.	Left Third Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	14	22	8	2	—	46
S. L.	8	104	108	20	26	266
L. L.	1	16	108	10	78	213
C.	—	—	6	—	7	13
W.	—	—	14	8	48	70
Totals	23	142	244	40	159	608

TABLE LIII.

Left Third Finger and Right Third Finger.

Right Third Finger.	Left Third Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	8	3	—	1	—	12
S. L.	14	91	30	3	2	140
L. L.	1	29	129	9	21	189
C.	—	8	14	6	5	33
W.	—	11	68	21	130	230
Totals	23	142	241	40	158	604

TABLE LIV.

Left Third Finger and Right Fourth Finger.

Right Fourth Finger.	Left Third Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	10	3	1	—	—	14
S. L.	11	110	71	11	21	224
L. L.	2	23	152	23	76	276
C.	—	—	2	4	9	15
W.	—	4	12	2	52	70
Totals	23	140	238	40	158	599

TABLE LV.

Left Fourth Finger and Right Thumb.

Right Thumb.	Left Fourth Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	8	4	5	—	—	17
S. L.	5	59	22	—	4	90
L. L.	6	112	121	3	16	258
C.	—	7	20	1	6	34
W.	—	39	121	5	22	187
Totals	19	221	289	9	48	586

TABLE LVI.

Left Fourth Finger and Right First Finger.

Right First Finger.	Left Fourth Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	14	45	20	—	2	81
S. L.	5	116	76	2	7	206
L. L.	—	32	69	1	9	111
C.	—	7	25	1	2	35
W.	—	21	100	5	28	154
Totals	19	221	290	9	48	587

TABLE LVII.

Left Fourth Finger and Right Second Finger.

Right Second Finger.	Left Fourth Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	10	22	13	—	—	45
S. L.	9	133	99	5	11	257
L. L.	—	55	124	3	25	207
C.	—	3	8	—	2	13
W.	—	10	46	1	10	67
Totals	19	223	290	9	48	589

TABLE LVIII.

Left Fourth Finger and Right Third Finger.

Right Third Finger.	Left Fourth Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	7	2	3	—	—	12
S. L.	8	104	25	—	3	140
L. L.	—	77	101	2	4	184
C.	1	6	20	2	—	29
W.	2	33	138	5	41	219
Totals	18	222	287	9	48	584

TABLE LIX.

Left Fourth Finger and Right Fourth Finger.

Right Fourth Finger.	Left Fourth Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	8	4	1	—	—	13
S. L.	11	164	41	—	3	219
L. L.	—	45	203	5	14	267
C.	—	2	10	2	1	15
W.	—	2	31	2	30	65
Totals	19	217	286	9	48	579

TABLE LX.

Right Thumb and Right First Finger.

Right First Finger.	Right Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	12	25	25	6	14	82
S. L.	4	58	109	9	35	215
L. L.	1	6	61	6	42	116
C.	1	—	11	3	22	37
W.	—	6	64	11	83	164
Totals	18	95	270	35	196	614

TABLE LXI.

Right Thumb and Right Second Finger.

Right Second Finger.	Right Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	9	18	12	3	4	46
S. L.	6	63	130	8	63	270
L. L.	1	12	107	17	79	216
C.	1	1	2	2	6	12
W.	—	1	20	5	44	70
Totals	17	95	271	35	196	614

TABLE LXII.

Right Thumb and Right Third Finger.

Right Third Finger.	Right Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	7	3	3	—	—	13
S. L.	5	48	60	5	22	140
L. L.	3	22	105	8	53	191
C.	—	7	16	3	7	33
W.	2	14	84	19	114	233
Totals	17	94	268	35	196	610

TABLE LXIII.

Right Thumb and Right Fourth Finger.

Right Fourth Finger.	Right Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	5	8	1	—	—	14
S. L.	8	61	120	4	32	225
L. L.	4	23	122	20	110	279
C.	—	—	3	3	8	14
W.	1	2	21	7	40	71
Totals	18	94	267	34	190	603

TABLE LXIV.

Right First Finger and Right Second Finger.

Right Second Finger.	Right First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	34	8	3	—	1	46
S. L.	38	154	33	12	35	272
L. L.	8	51	62	18	76	215
C.	1	2	5	—	5	13
W.	1	1	14	6	47	69
Totals	82	216	117	36	164	615

TABLE LXV.

Right First Finger and Right Third Finger.

Right Third Finger.	Right First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	10	3	—	—	—	13
S. L.	42	81	12	2	4	141
L. L.	12	78	47	12	41	190
C.	4	11	10	3	5	33
W.	13	42	46	20	113	234
Totals	81	215	115	37	163	611

TABLE LXVI.

Right First Finger and Right Fourth Finger.

Right Fourth Finger.	Right First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	12	2	—	—	—	14
S. L.	49	120	34	7	16	226
L. L.	17	81	64	23	95	280
C.	1	2	3	4	4	14
W.	3	9	11	2	46	71
Totals	82	214	112	36	161	605

TABLE LXVII.

Right Second Finger and Right Third Finger.

Right Third Finger.	Right Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	10	3	—	—	—	13
S. L.	24	100	16	1	—	141
L. L.	6	88	84	3	11	192
C.	4	15	11	1	1	32
W.	2	64	102	8	58	234
Totals	46	270	213	13	70	612

TABLE LXVIII.

Right Second Finger and Right Fourth Finger.

Right Fourth Finger.	Right Second Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	10	4	—	—	—	14
S. L.	28	142	47	3	6	226
L. L.	6	103	127	5	39	280
C.	—	4	6	2	3	15
W.	1	14	31	2	22	70
Totals	45	267	211	12	70	605

TABLE LXIX.

Right Third Finger and Right Fourth Finger.

Right Fourth Finger.	Right Third Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	7	5	1	—	1	14
S. L.	5	103	69	11	36	224
L. L.	1	29	111	16	120	277
C.	—	—	1	4	10	15
W.	—	1	5	1	63	70
Totals	13	138	187	32	230	600

TABLE LXX*.

Siblings' Left Thumb and Left First Finger.

Left First Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	17	36	41	9	7	110
S. L.	20	70	123	22	35	270
L. L.	6	28	73	17	27	151
C.	—	1	15	6	8	30
W.	6	17	60	22	53	158
Totals	49	152	312	76	130	719

* Tables LXX—LXXXVI are cross-hereditary tables, e.g. Left Thumb in First, Left First Finger in Second Sibling.

TABLE LXXI.

Siblings' Left Thumb and Left Second Finger.

Left Second Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	12	32	32	10	8	94
S. L.	19	65	123	26	45	278
L. L.	16	46	124	28	43	257
C.	1	1	3	1	8	14
W.	1	10	29	10	28	78
Totals	49	154	311	75	132	721

TABLE LXXII.

Siblings' Left Thumb and Left Third Finger.

Left Third Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	3	9	11	3	3	29
S. L.	21	49	74	14	21	179
L. L.	16	57	141	33	51	298
C.	2	7	15	4	10	38
W.	7	31	72	20	47	177
Totals	49	153	313	74	132	721

TABLE LXXIII.

Siblings' Left Thumb and Left Fourth Finger.

Left Fourth Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	2	12	9	1	1	25
S. L.	31	69	131	21	31	283
L. L.	11	58	142	40	75	326
C.	—	3	7	1	3	14
W.	3	7	16	8	16	50
Totals	47	149	305	71	126	698

TABLE LXXIV.

Siblings' Left Thumb and Right Thumb.

Right Thumb.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	—	11	9	2	2	24
S. L.	21	36	37	9	10	113
L. L.	18	71	172	25	45	331
C.	2	7	16	4	4	33
W.	8	29	79	35	71	222
Totals	49	154	313	75	132	723

TABLE LXXV.

Siblings' Left Thumb and Right First Finger.

Right First Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	11	40	35	12	4	102
S. L.	21	59	112	25	44	261
L. L.	7	26	61	13	23	130
C.	2	6	19	2	18	47
W.	8	23	86	23	43	183
Totals	49	154	313	75	132	723

TABLE LXXVI.

Siblings' Left Thumb and Right Second Finger.

Right Second Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	6	15	23	6	2	52
S. L.	27	81	147	25	45	325
L. L.	11	43	109	30	53	246
C.	—	2	3	3	6	14
W.	5	13	31	10	25	84
Totals	49	154	313	74	131	721

TABLE LXXVII.

Siblings' Left Thumb and Right Third Finger.

Right Third Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	2	8	5	1	—	16
S. L.	23	57	68	15	14	177
L. L.	11	39	127	27	44	248
C.	1	11	12	5	14	43
W.	12	38	102	26	60	238
Totals	49	153	314	74	132	722

TABLE LXXVIII.

Siblings' Left Thumb and Right Fourth Finger.

Right Fourth Finger.	Left Thumb. A.	S. L.	L. L.	C.	W.	Totals
A.	3	5	9	1	1	19
S. L.	29	72	129	24	31	285
L. L.	14	60	129	36	75	314
C.	—	4	8	2	5	19
W.	3	8	33	12	18	74
Totals	49	149	308	75	130	711

TABLE LXXIX.

Siblings' Left First Finger and Left Second Finger.

Left Second Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	35	39	6	4	9	93
S. L.	47	119	62	4	47	279
L. L.	24	96	61	13	62	256
C.	2	2	2	1	7	14
W.	2	16	21	7	32	78
Totals	110	272	152	29	157	720

TABLE LXXX.

Siblings' Left First Finger and Left Third Finger.

Left Third Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	11	15	2	1	—	29
S. L.	45	75	32	4	22	178
L. L.	31	117	70	11	67	296
C.	7	11	5	—	15	38
W.	16	51	43	14	52	176
Totals	110	269	152	30	156	717

TABLE LXXXI.

Siblings' Left First Finger and Left Fourth Finger.

Left Fourth Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	11	9	3	1	1	25
S. L.	55	126	57	5	40	283
L. L.	36	113	74	17	83	323
C.	—	5	4	1	4	14
W.	3	9	13	4	20	49
Totals	105	262	151	28	148	694

TABLE LXXXII.

Siblings' Left First Finger and Right Thumb.

Right Thumb.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	6	12	3	—	2	23
S. L.	35	47	13	2	16	113
L. L.	45	133	77	14	63	332
C.	6	11	7	1	8	33
W.	19	69	51	13	69	221
Totals	111	272	151	30	158	722

TABLE LXXXIII.

Siblings' Left First Finger and Right First Finger.

Right First Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	36	42	8	4	9	99
S. L.	36	125	56	8	38	263
L. L.	14	50	26	6	33	129
C.	7	13	13	2	12	47
W.	18	40	49	10	66	183
Totals	111	270	152	30	158	721

TABLE LXXXIV.

Siblings' Left First Finger and Right Second Finger.

Right Second Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	16	22	8	1	4	51
S. L.	66	141	60	9	51	327
L. L.	22	82	65	13	63	245
C.	1	2	4	—	6	13
W.	6	24	15	7	31	83
Totals	111	271	152	30	155	719

TABLE LXXXV.

Siblings' Left First Finger and Right Third Finger.

Right Third Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	9	7	1	—	—	17
S. L.	46	81	29	3	18	177
L. L.	26	98	60	6	58	248
C.	2	18	7	1	13	41
W.	28	66	56	19	68	237
Totals	111	270	153	29	157	720

TABLE LXXXVI.

Siblings' Left First Finger and Right Fourth Finger.

Right Fourth Finger.	Left First Finger. A.	S. L.	L. L.	C.	W.	Totals
A.	9	7	—	—	3	19
S. L.	57	124	62	11	32	286
L. L.	32	116	65	15	84	312
C.	3	5	3	2	6	19
W.	6	16	20	2	29	73
Totals	107	268	150	30	154	709

TABLE LXXXVII.

Grandparent's Left First Finger and Grandchild's Left First Finger.

Grandchild.	Grandparent. A.	S. L.	L. L.	C.	W.	Totals
A.	9	31	—	2	23	65
S. L.	35	82	19	1	63	200
L. L.	8	22	7	—	18	55
C.	1	6	3	—	5	15
W.	12	18	9	—	31	70
Totals	65	159	38	3	140	405

TABLE LXXXVIII.

Grandparent's Right First Finger and Grandchild's Right First Finger.

Grandchild.	Grandparent. A.	S. L.	L. L.	C.	W.	Totals
A.	17	23	10	—	27	77
S. L.	37	61	21	1	44	164
L. L.	6	14	13	4	18	55
C.	5	10	2	—	8	25
W.	19	24	9	2	30	84
Totals	84	132	55	7	127	405

TABLE LXXXIX.

Uncle's First Left Finger and Nephew's First Left Finger.

Nephew.	Uncle. A.	S. L.	L. L.	C.	W.	Totals
A.	21	39	4	3	8	75
S. L.	44	77	20	16	28	185
L. L.	11	32	14	11	23	91
C.	3	13	4	6	11	37
W.	10	31	23	13	28	105
Totals	89	192	65	49	98	493

TABLE XC.

Uncle's First Right Finger and Nephew's First Right Finger.

Nephew.	Uncle. A.	S. L.	L. L.	C.	W.	Totals
A.	9	38	8	2	3	60
S. L.	16	94	16	9	46	181
L. L.	11	38	10	3	27	89
C.	1	19	10	3	13	46
W.	9	39	13	15	41	117
Totals	46	228	57	32	130	493

TABLE XCI.

Uncle's First Left Finger and Niece's First Left Finger.

Niece.	Uncle. A.	S. L.	L. L.	C.	W.	Totals
A.	18	31	7	2	19	77
S. L.	41	99	23	25	51	239
L. L.	4	18	11	13	22	68
C.	7	10	5	6	10	38
W.	7	33	13	8	10	71
Totals	77	191	59	54	112	493

TABLE XCII.

Uncle's First Right Finger and Niece's First Right Finger.

Niece.	Uncle. A.	S. L.	L. L.	C.	W.	Totals
A.	10	43	10	4	22	89
S. L.	13	98	20	12	45	188
L. L.	4	19	15	9	23	70
C.	5	25	8	4	17	59
W.	8	35	8	9	27	87
Totals	40	220	61	38	134	493

TABLE XCIII.

Aunt's First Left Finger and Nephew's First Left Finger.

Nephew.	Aunt. A.	S. L.	L. L.	C.	W.	Totals
A.	5	25	8	4	11	53
S. L.	24	91	28	20	29	192
L. L.	9	44	17	9	16	95
C.	3	20	4	14	4	45
W.	7	50	39	20	13	129
Totals	48	230	96	67	73	514

TABLE XCIV.

Aunt's First Right Finger and Nephew's First Right Finger.

Nephew.	Aunt. A.	S. L.	L. L.	C.	W.	Totals
A.	10	20	5	6	12	53
S. L.	28	89	28	25	29	199
L. L.	3	33	13	7	12	68
C.	6	23	10	9	15	63
W.	10	57	7	21	36	131
Totals	57	222	63	68	104	514

TABLE XCV.

Aunt's First Left Finger and Niece's First Left Finger.

Niece. \ Aunt.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	23	8	17	17	77
S. L.	24	139	28	29	44	264
L. L.	—	25	19	8	11	63
C.	4	10	7	8	10	39
W.	4	37	13	17	17	88
Totals	44	234	75	79	99	531

TABLE XCVI.

Aunt's First Right Finger and Niece's First Right Finger.

Niece. \ Aunt.	A.	S. L.	L. L.	C.	W.	Totals
A.	17	41	10	9	20	97
S. L.	24	103	27	27	31	212
L. L.	8	30	16	7	15	76
C.	5	18	9	10	16	58
W.	9	32	14	12	21	88
Totals	63	224	76	65	103	531

TABLE XCVII.

Male Cousins' First Left Finger.

Second Cousin. \ First Cousin.	A.	S. L.	L. L.	C.	W.	Totals
A.	48	96	39	26	56	265
S. L.	96	148	69	53	129	495
L. L.	39	69	38	28	75	249
C.	26	53	28	8	44	159
W.	56	129	75	44	118	422
Totals	265	495	249	159	422	1590

TABLE XCVIII.

Male Cousins' First Right Finger.

Second Cousin. \ First Cousin.	A.	S. L.	L. L.	C.	W.	Totals
A.	52	100	19	13	36	220
S. L.	100	230	65	50	159	604
L. L.	19	65	38	13	61	196
C.	13	50	13	16	35	127
W.	36	159	61	35	152	443
Totals	220	604	196	127	443	1590

TABLE XCIX.

Female Cousins' First Left Finger.

Second Cousin. \ First Cousin.	A.	S. L.	L. L.	C.	W.	Totals
A.	26	78	27	17	35	183
S. L.	78	294	104	44	84	604
L. L.	27	104	44	11	25	211
C.	17	44	11	14	24	110
W.	35	84	25	24	38	206
Totals	183	604	211	110	206	1314

TABLE C.

Female Cousins' First Right Finger.

Second Cousin. \ First Cousin.	A.	S. L.	L. L.	C.	W.	Totals
A.	30	76	26	29	34	195
S. L.	76	206	65	77	72	496
L. L.	26	65	36	29	34	190
C.	29	77	29	26	35	196
W.	34	72	34	35	62	237
Totals	195	496	190	196	237	1314

TABLE CI.

Male and Female Cousins' First Left Finger.

Female Cousin. \ Male Cousin.	A.	S. L.	L. L.	C.	W.	Totals
A.	27	68	37	23	58	213
S. L.	115	247	114	64	153	693
L. L.	24	66	42	16	69	217
C.	22	36	23	14	32	127
W.	41	86	40	29	64	260
Totals	229	503	256	146	376	1510

TABLE CII.

Male and Female Cousins' First Right Finger.

Female Cousin. \ Male Cousin.	A.	S. L.	L. L.	C.	W.	Totals
A.	26	63	38	12	65	204
S. L.	93	228	77	50	126	574
L. L.	23	78	33	22	55	211
C.	32	94	25	22	62	235
W.	30	97	40	28	91	286
Totals	204	560	213	134	399	1510

THE INHABITANTS OF THE EASTERN MEDITERRANEAN.

By L. H. DUDLEY BUXTON, M.A.
(Department of Anatomy, Oxford.)

THE ethnology of the Eastern Mediterranean was in ancient times a matter of keen interest to the Greeks themselves; since the revival of Greek learning it has concerned many scholars. Unfortunately owing, no doubt, to the wealth of cultural and documentary evidence, the physical affinities have been rather neglected and the earlier excavators either reburied or threw away the bones they discovered.

Professor J. L. Myres was good enough to take me as his assistant to Cyprus in 1913, and I was thus given an opportunity of studying the anthropology of that island at first hand. The evidence brought forward in this paper has been collected while studying my Cypriot material and is a natural corollary to work in Cyprus. I am indebted to Professor Myres and to Professor Arthur Thomson for continual advice and criticism during the progress of my work at Oxford, and to Professor Karl Pearson for initiating me into biometric methods. The photographs of crania were taken by Mr W. Chesterman, Assistant in the Anatomical Department, Oxford.

The method followed in this paper has been first to enumerate a few of the representative views on the population of the Eastern Mediterranean, secondly to discuss in detail the cephalic index, glabello-occipital length, greatest head breadth, upper facial index, nasal index, stature and pigmentation, to which characters we are limited by our lack of further evidence, and thirdly to attempt to summarize the tentative conclusions arrived at in dealing with racial problems in the Eastern Mediterranean.

While it is generally admitted by most writers that the population of the Eastern Mediterranean consists of an admixture of Mediterranean and Alpine—or alternatively Armenoid types, extremely diverse opinions have been held both about the original population and also about the degree of admixture which has taken place. v. Luschan in a paper of 1891 [1] when dealing with the early population of Lycia, declares that "there are clearly two types, one short headed found especially in the mountains and in the swamps and a second long headed found in the towns and on the coast, the latter probably do not represent a single unit. It would be difficult to bring the figures especially the high indices of Makri, Xanthos, Rekowa and Myra into line with those from the East Lycian coast." The cephalic indices of the Asiatic Greeks have two summits 75 and 88 with the lesser summit at 75. The broad high skulls belong to the old type who reach from the

Caucasus to the Euphrates. In a later paper[2] this view is expanded and v. Luschan expresses the belief that the earliest inhabitants of the Mediterranean were Armenoid and were the authors of the -issos, -andos, etc., terminations for place names. The Dorians were a kind of backwash of the same people going East instead of West. The long blond (!) skulls of the modern Kurds do not appear in Asia Minor before the second century B.C.; there is otherwise no trace of any other people, but the place names with the terminations in question do appear. He disagrees with Sergi, who believes that the Mediterranean race provided the earliest inhabitants of the Mediterranean, for he maintains that the Armenoid type appeared in Sicily and in Sardinia at the very end of the Neolithic period.

Ripley[3] in his survey of the peoples of Eastern Europe has three main divisions, Greek, Turk and Slav, the first, except in Thessaly, being essentially a littoral people, the last an inland people and the Turks a negligible factor. He states that all authorities agree that the ancient Hellenes were long-headed and of Mediterranean stock and quotes v. Luschan's curve. The modern Greeks are very brunet. In dealing with the people of Western Asia Ripley depends mainly on Chantre for his information. He calls attention to two types, Kurds and Armenians, the one dolichocephalic, the other brachycephalic. The Kurds are the oldest and have been quite unaffected by all invasions. The custom of shaping the heads of the children has accentuated the natural differences of these peoples. In regard to the question as to which is the earlier people, the Armenoid or the Mediterranean, Ripley maintains that v. Luschan's argument depends on the scattered nature of the Armenoid settlements and on their various religions; the long heads occur on the coast especially in Greek necropoles. Ripley believes that the Mediterranean people were the earliest arrivals, the Turks the latest. The Armenoids are of importance because they link up Europe and Asia, they are perhaps the Pelasgians. In Persia this great contrast disappears possibly owing to the great plains.

Sergi[4] states, "I am convinced that the primitive population of Lycia and the rest of Asia Minor as also of Syria is of the same type as the Egyptian and derived from the same centre of diffusion. This primitive population...could not have been of brachycephalic Armenoid type....It is probable that the immigrants encountered a population coming from Northern Arabia but as the skull characters of the two races were allied it is difficult to distinguish them."

Ridgeway[5] states that the original inhabitants of Greece were, as are the present inhabitants, a dark people. They already inhabited Greece in the Neolithic period. About 1500 B.C. the immigration of Teutonic people, the Achaeans, took place. The Illyrio-Thracian tribes belong to the same stock as the original race, but were conquered in many cases by Keltic from the Alps. The Dorians were like the Thessalians an Illyrian tribe, and Herodotus (VIII, 43) thought them Macedonian. The Spartans thought there was a difference of race between the Dorians and the Achaeans.

As far as physical characteristics are concerned the Homeric Achaeans were tall men with fair hair. There is no reference to the colour of the Spartans' hair so we

imagine that it did not differ from that of the rest of the Greeks; we should have expected a mention of it both when before Thermopylae the Spartans are combing their long hair and in the Lysistrata.

Deniker[27] in general deals with the modern population. He states[28], however, "en somme si les crânes grecs d'aujourd'hui sont tous sous-brachycéphales, tandis que les anciens crânes mesurés jusqu'à présent sont pour la plupart sous-dolichocéphales, il ne faut pas oublier que l'étude des crânes connus de la Grèce classique très peu nombreux et appartenant presque tous à l'Attique, n'autoriserait aucune conclusion générale, et que d'après les temoignages mêmes des auteurs anciens et des monuments de l'art grec une très grande partie de l'ancienne population de la Grèce était brachycéphale et surtout sous-brachycéphale." In regard to the modern population he considers those west of 20° E. of Paris to belong to his Adriatico-Albanian type of tall stature and brachycephalic heads while those east, the Greeks of Thessaly and perhaps those of Attica, are lower in stature and cephalic index.

Pittard[6] in his paper on the Greeks strongly disagrees with this conclusion.

Myres[7] sums up his views on the subject: "In the Eastern basin of the Mediterranean, the hold of the Mediterranean race on the whole north shore is of the slightest...and there is growing reason to believe that the strong 'Alpine' strain in the Morea, which is certainly ancient, may even be primitive there. Even in the Aegean islands, and in Crete, which were admittedly occupied early and decisively by Mediterranean man, traces of continental intruders, of Alpine affinities, begin already at the close of the Stone Age, shewing that Alpine man was already present in force on the neighbouring mainlands. Mediterranean man, therefore, must be regarded as in all probability an intruder from the south; just as 'Alpine' man reveals himself more and more clearly now as a longitudinal immigrant from the east along the Mountain Zone. Both movements alike are very ancient."

Material Used.

Neolithic Period.

The only authenticated Neolithic skull from the north coast of the Eastern Mediterranean is reported by Velde[8] from the island of Leukas. This skull, which was very fragmentary and has not been thoroughly described, differed from other skulls dating from the second millenium B.C. found in the same island in the following particulars. It was brachycephalic with an index of 81, and the cranial bones were markedly thick. As far as can be judged from the photographs and the brief description of the author this skull did not differ essentially from the Bronze Age skulls found at Lapithos in Cyprus. It may be an unusual Mediterranean type, but in view of the definite statement of the finder that it differed from the Mediterranean skulls found in the island and of later date, we may provisionally suggest that the same mixed type which we shall shew is so characteristic of the area under discussion in later times, may have also been found as early as the Neolithic Age, but at present we have insufficient evidence to justify such a conclusion.

Bronze Age.

Five skulls of this period, but not apparently very exactly dateable, were found in Leukas during Dorpfeld's excavation [8]. They were remarkable in their homogeneity, all having cephalic indices from 74—76. The cranial bones were reported to be noticeably thin, but no other details have been published.

We have a fairly complete series of Bronze Age skulls from Crete [2], [9], [10], [11], coming from Hagios Nikolaos, Palaikastro and Gournia.

The Palaikastro series numbered 64 (of which 46 were male). No Bronze Age skulls appear to have been excavated in Asia Minor. The Cypriot series from Lapithos number fifteen [18]. The contrast on which Velde has insisted so strongly between thick and thin bones of the cranial vault could be observed in the Bronze Age skulls at Lapithos extending over a series of about a hundred, which could not be measured, but both types were present often in the same tombs.

Early Iron Age.

The early Iron Age appears to be represented only by three skulls from Amathus in Cyprus excavated by Myres [12]. Two are now in the Department of Human Anatomy, University Museum, Oxford, and one was in 1913 in the Cyprus Museum.

Hellenic, Hellenistic and Roman.

There are nine skulls in the University Museum in Oxford, varying in date from early Hellenic to Hellenistic Four fourth century Athenians excavated by Schliemann were examined by Virchow [13]. There are seven skulls of this period in the Cyprus Museum and one of Roman date from Leukas.

In addition to the above there are four series of "ancient" Greek crania in the Williamson Collection, whose date is uncertain. They include six from Samos, four from Corfu, and two from Myrma on the coast of Asia Minor.

There is a small series, presumably of Roman date, from Halicarnassus in Caria also in the Williamson Collection, but unfortunately they were not sufficiently well preserved for accurate measurement.

Modern skulls whose geographical provenances are authenticated are rare. Weisbach [31] published a large series but from so wide an area that it has been unfortunately necessary to reject them. The only series available for statistical treatment come from Crete and Adalia [1] in Asia Minor.

Measurements on the living are available for statistical treatment from the following peoples. A small number of men were examined by Velde [8] in Leukas (St Maura), an island off the west coast of Greece just north of the Gulf of Corinth. Schiff [14] measured 99 men from Mani on the Southern Peloponnese and a small series from Meligala in Messenia. A considerably larger series of Albanians were examined by Haberlandt and Lebzelter [15]. Unfortunately these latter came from a rather large area. Measurements were made in Crete by Hawes, v. Luschan and Schiff [16]. The first has unfortunately not yet published his figures. v. Luschan is the author of several exceedingly interesting papers on Lycia. The Cypriot

measurements were made by the author. The figures used for comparative purposes have been drawn from the following sources: for Sardinia and Corsica, Duckworth [10], for Egypt, Craig [17].

Duckworth and Craig have reduced their own material to statistical form and Duckworth has published Hawes' figures.

Cultural conditions are of special importance in regard to the Bektasch and Tadchadsky, Lycian gypsies, who contrast very strongly with their Greek and Turkish neighbours not only in their measurements but what is still more important the standard deviations of these measurements. These people form a small endogamous religious community which claims to keep itself free from intermixture with either Christian Greeks or Moslems, a claim substantiated by the low standard deviations. Similar communities are found among the Druses and Maronites on Mount Lebanon and to a lesser degree among the Jews. The purity of the Lycian gypsies will be seen to be specially striking because they live amongst an unusually mixed population.

The measurements from Cyprus and Crete have been subdivided according to localities. In Crete the eparchies of Selinos and Sphakia, selected by v. Luschan as representing a special "type," have been worked out separately; in Cyprus [18] I have divided my own measurements into four groups. Group I is a miscellaneous group mostly from Nicosia and the neighbourhood, Group II the villages on the north coast, Group III the villages round the Bay of Salamis on the east coast and Group IV the village of Levkoniko in the Mesaoria, this division being purely geographical. In both cases the small subdivisions are included in a final summation which I have termed in the case of Cyprus "groups combined" and in Crete, "whole island." Schiff's measurements for Pyrgiótissa, which include nine men from Sphakia, have not been included in the Cretan total which is limited to v. Luschan's 320 men.

For the physical characters of the Jews a good deal of material is available. This has been very conveniently summarized by Fishberg [19], who includes, besides his own observations (a series of 500 measured in New York), series from Weissenberg [20], Lombroso [21], Blechman [22], Yakowenko [23] and others. Schiff [24] has summarized some figures and has treated them statistically; unfortunately most of his series which include Askenazim, Shephardim, Grusinians, and Jews from the Yemen, Aleppo and Morocco are except for the first two small, his probable errors have also been wrongly calculated. Weissenberg has published a number of valuable papers on the Jews. Use has been made of his figures from S. Russia [25], the Yemen [25], and Samarkand [26].

Cephalic Index.

The cephalic indices of living adult males have been tabulated in Table I. Measurements from Corsica and two Egyptian series have been added for comparative purposes. None of the means appear to be in agreement with the pure Mediterranean type represented in Corsica and in a less pure form in Egypt. If the means are

compared by taking the difference between any two and the probable error of that difference and we accept the difference as being significant where it is greater than three times its probable error, we find that on v. Luschan's figures there is no

TABLE I.

Cephalic Index. Living Adult Males.

Provenance	Number	Mean	σ	V
Corsica	53	75·53 ± ·27	2·90 ± ·19	3·82 ± ·25
Albania	140	86·07 ± ·25	4·44 ± ·17	5·16 ± ·23
Leukas	42	84·55 ± ·40	3·86 ± ·28	4·60 ± ·34
Meligala	33	82·49 ± ·38	3·21 ± ·26	3·89 ± ·32
Mani	99	79·92 ± ·26	3·77 ± ·18	4·72 ± ·23
Crete (Hawes)	1600	79·20 ± ·07	4·10 ± ·05	5·17 ± ·06
„ (Schiff) Pyrgiótissa...	64	79·27 ± ·32	3·89 ± ·23	4·91 ± ·29
„ (v. Luschan) Selinos and Sphakia	51	81·68 ± ·37	3·91 ± ·26	4·80 ± ·32
„ Whole island ...	320	79·26 ± ·16	4·24 ± ·11	5·35 ± ·14
Lycia, Turks	187	81·29 ± ·26	5·21 ± ·18	6·41 ± ·22
„ Greeks	179	80·27 ± ·35	6·94 ± ·25	8·65 ± ·31
„ Gypsies	53	87·51 ± ·26	2·83 ± ·19	3·23 ± ·21
Cyprus, Group I	111	81·41 ± ·26	3·99 ± ·18	4·91 ± ·22
„ „ II	221	81·94 ± ·13	4·02 ± ·13	4·90 ± ·16
„ „ III	167	83·38 ± ·24	4·63 ± ·17	5·55 ± ·20
„ „ IV	87	84·16 ± ·31	4·25 ± ·22	5·03 ± ·26
„ Groups combined	586	82·54 ± ·11	4·09 ± ·08	4·96 ± ·38
Alexandria	643	76·29 ± ·08	3·13 ± ·06	4·10 ± ·08
Giza	326	74·75 ± ·11	2·83 ± ·08	3·79 ± ·10

significant difference between the Lycian Greeks and the Cretans although Hawes' figures, a very much larger series, suggest a possible difference; the Lycians and Maniots are very similar, there is a significant difference between the Lycians and both the men of Meligala and of Cyprus and a very marked difference between Cretans and Cypriots.

TABLE II.

Differences between Cephalic Indices.

People	Compared	Difference	Remarks
Lycian Greeks ...	Cretans	1·01 ± ·38	v. Luschan's figures
„ „ ...	„	1·07 ± ·35	Hawes' figures
„ „ ...	Maniots	·35 ± ·44	—
„ „ ...	Meligala	2·45 ± ·51	—
„ „ ...	Cypriots	2·27 ± ·37	—
Cretans	„	3·34 ± ·13	Hawes' figures

It would appear from Hawes' figures that Lycians and Cretans possibly did not belong to the same group, but the excess of the difference over three times the

error is so small that for the moment the matter can be left open. Grouping our mean values we have three classes:

Under 81	82	Over 84·5
Crete	Meligala	Leukas
Mani	Cyprus	Albania
Lycia (Greeks and Turks)		Lycian Gypsies

That such a grouping is of any real significance seems unlikely when we turn to the standard deviations. Here we have a striking contrast between Lycian Greeks and all the other Greek countries, suggesting a very big mixture of two races. Apart from the figure from Meligala in Messenia where the numbers measured are too small for any satisfactory review of the standard deviation it will be seen that in every case we have a much greater standard deviation than in the comparatively homogeneous Corsica and greater than in the cosmopolitan Alexandria. It should also be noticed that the smaller series have smaller standard deviations suggesting that the small areas from which the men were drawn are more homogeneous than the larger areas as for instance Crete or Cyprus. The evidence of four areas measured in Cyprus does not however bear this statement out, as the difference between the Cypriot standard deviations either between the villages themselves, or between the villages and the total number of men measured, is seldom of any significance. It may be concluded that we are dealing with a very mixed population. This fact is strikingly confirmed by two pieces of evidence. First, the local differences between different groups of villages in Cyprus. The villages on the north coast (221 adult males) have a cephalic index of 81·94 and 167 adult males from villages round Enkomi one of 83·38. Secondly, 60 adult males from five selected eparchies of Crete have a cephalic index of 76·6, and 72 adult males from another five eparchies a cephalic index of 81·8. The selection of eparchies by v. Luschan on the basis of cephalic index naturally increases the difference and tends to emphasize the third piece of evidence more strongly than it perhaps deserves. Even if we disregard it however there appears a good case for a great deal of admixture among the living "Greeks" and at the same time considerable local variation and the appearance of similar types at opposite ends of the Greek world.

One further point which tends to suggest mixture and proves the necessity of taking large series is the great contrast between Hawes' and v. Luschan's figures for Sphakia and Selinos. The cephalic index for the whole series of Cretans is the same in both cases and differs by only ·06 per cent., whereas however v. Luschan believes that the cephalic index of the Sphakiots (30 measured) and Selinots (21 measured) to be 81·6 in both cases. Hawes, on the evidence of 98 and 35 respectively, has a cephalic index of 85. The very close resemblance of their final mean suggests a similar technique, but this immense local divergence shews clearly the danger of elaborating racial theories on an insufficient number of cases. It also throws very considerable doubt on v. Luschan's cephalic index map of Crete and any conclusions that may be drawn from it.

While the high standard deviations of Lycia are noticeable, the extremely low standard deviation of the Lycian gypsies calls for special comment. We have already drawn attention to the fact that these people formed a strictly endogamous community and did not mix with their neighbours. The evidence of the low standard deviation, especially in contrast with the great variation shewn by people living in the same place but belonging to a different social *milieu*, supports the claim of these gypsies to be a pure blooded race.

The crania (see Table III) belong to very small series and it is difficult to argue with certainty. It appears that as a general rule the modern Greeks are slightly

TABLE III.

Cephalic and Cranial Indices.

Provenance	Adult Living Males		Modern Crania			Ancient Crania			
	No.	Cephalic Index	No.	Sex	Cranial Index +2 units	Date	No.	Sex	Cranial Index +2 units
Leukas ...	42	84·55	—	—	—	Neolithic	1	?	83
—	—	—	—	—	—	Bronze Age	5	—	75
—	—	—	—	—	—	Late Roman	1	—	77
Corfu ...	—	—	—	—	—	Ancient	4	—	79·98
Crete ...	1600	79·20	52	♂	80	"Beginning of Bronze Age"...	6	—	77·1
—	—	—	17	♀	82·9	Early Middle Minoan ...	46	♂	75·10
Crete, Khania	—	—	13	♂	77·8	Middle Minoan	13	—	77·5
—	—	—	—	—	—	Early Late Minoan	5	—	78·5
—	—	—	—	—	—	Late Minoan III	7	—	81·1
Lycia (Greeks)	179	80·27	93	—	79·9	—	—	—	—
Samos ...	—	—	—	—	—	"Ancient"	6	♂	79·40
Cyprus ...	586	82·54	—	—	—	Bronze Age	14	—	80·35
—	—	—	—	—	—	Hellenic and Roman ...	7	—	79·20
Athens ...	60(?)	79·6	—	—	—	Early Hellenic to Hellenistic	13	—	79·38

more brachycephalic than the ancient skulls from the small place, even after making allowance for the difference between the measurements of the living and the dead. This difference appears still more striking when the distribution into dolichocephalic, etc. is made. Even granting however that this difference exists, considering the small number of crania, it is perhaps more remarkable that the ancient skulls should be so extremely close considering the very large standard deviations which we have found. The comparison has been made in Table IV.

There are however certain points of resemblance and difference which need special attention. First there is in general a closer approximation between the earliest cranial indices and modern ones than among those of later date. This holds true in every case. Secondly there is only one case of a considerable difference between the ancient crania and the modern. This occurs in Leukas. The one early (Neolithic) skull has a cranial index of 81, and this index is not very different from that of the modern men of Leukas. The evidence of the five later skulls with

their apparent great homogeneity—for details have not yet been published—seems to suggest a possibility of some change in Bronze Age times.

TABLE IV.

Comparison of Ancient and Modern Cephalic Indices.

Bronze Age. Locality	Cranial Index +2	Modern Living Males. Locality	Cephalic Index	Difference and Remarks	Whether difference significant
Lapithos	80·35 ± ·69	Lapithos	81·94 ± ·18	1·59 ± ·71	No
,,	,,	Enkomi	83·34 ± ·23	2·99 ± ·73*	Yes
,,	,,	Cyprus	82·54 ± ·11	2·19 ± ·70	No?

* The difference between the modern inhabitants of the Lapithos and Enkomi areas is 1·40 ± ·29.

Only one ancient series is available for statistical treatment, Duckworth's series from Palaikastro. The figures are:

No.	Sex	*M*	σ	*V*
46	♂	73·10 ± ·38	4·03 ± ·28	5·5 ± ·39

These figures seem to suggest that the mixing of racial elements in Crete had reached in the Early Middle Minoan Period a similar state to modern times; a small series of skulls from Hagios Nikolaos, dated by Tod at the very beginning of the Bronze Age, also present the appearance of a mixed population. It has been argued that a change took place in the population of Crete in Minoan times. Such evidence as we have at present would seem to suggest that the mixed character of the population had already been established and that the variations which appear in the cephalic indices of the slender number of skulls at our disposal are such that might be expected in such a mixed population. We are inclined to believe that a similar early hybridisation had taken place in Cyprus though our numbers are insufficient for statistical treatment.

Let us next consider the two factors which make up the cranial index, namely, the glabello-occipital length (Table V) and the greatest head breadth (Table VI). The data are less than in cephalic indices as in many cases a complete series of measurements do not appear to have been published. There are several points of interest that appear from an examination of the data. First that in Cyprus the head breadth is very stable and the differences in cephalic index are due to variations in the glabello-occipital length, secondly, in Crete the differences in cephalic index are due to variations in the head breadth, the length being stable. The whole of the Lycian figures have not been published but two small groups of Bektasch and Tadchadsky have an exactly similar head breadth, the difference in length causing the variation in the index. The variation in cephalic index from the south to the north of the Peloponnese would appear to be due to both a shortening and a broadening of the head, but our figures are not sufficient to be conclusive. In spite of the fact that neither the Cypriot nor the Cretan indices

TABLE V.

Glabello-Occipital Length. Living Adult Males.

Provenance	Number	Mean	σ	V
Leukas	42	182·8 ± ·66	6·34 ± ·47	3·47 ± ·26
Meligala	33	185·38 ± ·57	4·88 ± ·41	2·63 ± ·22
Mani	99	187·09 ± ·41	6·01 ± ·29	3·32 ± ·16
Crete, Pyrgiótissa	64	186·06 ± ·57	6·80 ± ·41	3·67 ± ·22
„ Selinos and Sphakia	51	190·95 ± ·69	7·59 ± ·51	3·98 ± ·27
„ Whole island ...	320	191·76 ± ·52	7·24 ± ·19	3·78 ± ·10
Lycian Gypsies	53	176·33 ± ·53	5·73 ± ·38	3·25 ± ·21
Cyprus, Group I	111	183·12 ± ·40	6·18 ± ·28	3·38 ± ·15
„ „ II	221	182·41 ± ·31	6·90 ± ·22	3·78 ± ·12
„ „ III	167	178·43 ± ·34	6·42 ± ·24	3·60 ± ·13
„ „ IV	87	178·29 ± ·47	6·50 ± ·33	3·65 ± ·19
„ Groups combined	586	180·81 ± ·20	7·00 ± ·14	3·87 ± ·08
Alexandria	643	189·74 ± ·16	5·99 ± ·11	3·16 ± ·06
Giza	326	191·66 ± ·22	5·75 ± ·15	3·00 ± ·08

TABLE VI.

Head Breadths. Living Adult Males.

Provenance	Number	Mean	σ	V
Leukas	42	154·11 ± ·55	5·24 ± ·39	3·40 ± ·25
Meligala	33	152·59 ± ·67	5·72 ± ·47	3·75 ± ·31
Mani	99	150·99 ± ·35	5·15 ± ·25	3·43 ± ·18
Crete, Pyrgiótissa	64	147·91 ± ·51	6·03 ± ·36	4·08 ± ·25
„ Selinos and Sphakia	51	157·21 ± ·48	5·11 ± ·34	3·25 ± ·22
„ Whole island ...	320	151·76 ± ·23	6·07 ± ·16	4·00 ± ·11
Lycian Gypsies	53	150·67 ± ·38	4·12 ± ·27	2·73 ± ·18
Cyprus, Group I	111	148·76 ± ·38	5·99 ± ·27	3·38 ± ·15
„ „ II	222	149·24 ± ·24	5·19 ± ·17	3·48 ± ·11
„ „ III	167	148·54 ± ·28	5·39 ± ·20	3·63 ± ·13
„ „ IV	87	149·63 ± ·38	5·22 ± ·27	3·49 ± ·18
„ Groups combined	587	149·10 ± ·15	5·35 ± ·11	3·59 ± ·07

are the least, the Cypriots have the narrowest heads recorded and the Cretans the longest.

With these facts in mind it might have been conjectured that the coefficients of variation of the variable factor would be greater than the coefficients of variation of the more constant factor. This however does not prove to be the case. In no case in our series (see Table VII) is the difference between the coefficients of variation of any significance. The head does appear to have a slightly more variable length among brachycephalic people, and to have a slightly more variable breadth among dolichocephalic people, but statistical evidence is in our present series against such a conclusion being definitely drawn.

TABLE VII.

Differences between Coefficients of Variation. Head Lengths and Breadths.

Locality	Difference	Locality	Difference
Leukas	+ ·07 ± ·36	Lycian Gypsies ...	+ ·52 ± ·28
Meligala	− 1·12 ± ·38	Cyprus, Group I ...	·00 ± ·21
Mani	− ·11 ± ·24	„ „ II ...	+ ·30 ± ·16
Crete, Pyrgiótissa	− ·41 ± ·34	„ „ III ...	− ·03 ± ·18
„ Selinos and Sphakia	+ ·73 ± ·35	„ „ IV ...	+ ·16 ± ·26
„ Whole island ...	− ·22 ± ·15	Groups combined	+ ·28 ± ·11

+ indicates that the coefficient of variation of the head length is greater, − that that of the head breadth is greater.

Upper Facial Index.

The upper facial index has been reduced for a small series (see Table VIII), but after inspection the conclusion I arrived at was that on the whole as a racial test

TABLE VIII.

Upper Facial Index. Living Adult Males.

Provenance	Number	Mean	σ	V
Leukas	42	52·76 ± ·30	3·09 ± ·21	5·85 ± ·38
Meligala	33	56·85 ± ·47	4·02 ± ·33	7·08 ± ·59
Mani	98	57·45 ± ·25	3·73 ± ·18	6·48 ± ·31
Crete, Selinos and Sphakia ...	49	53·00 ± ·28	2·95 ± ·20	5·56 ± ·37
Albania	138	48·42 ± ·18	3·20 ± ·13	6·61 ± ·27
Lycian Gypsies	53	54·23 ± ·29	2·88 ± ·22	5·31 ± ·40
Cyprus, Group I	111	50·10 ± ·25	3·92 ± ·18	7·82 ± ·35
„ „ II	221	48·63 ± ·12	2·69 ± ·09	5·53 ± ·18
„ „ III	167	49·71 ± ·19	3·61 ± ·13	7·26 ± ·27
„ „ IV	87	50·90 ± ·28	3·87 ± ·20	7·60 ± ·39
Groups combined ...	586	49·52 ± ·10	3·71 ± ·07	7·50 ± ·15

this index did not appear to be of great value, at least within the area under consideration. The reasons for this conclusion are that the indices fall into three groups:

(1) Meligala and Mani.

(2) Crete, Lycia and Leukas.

(3) Cyprus.

We have previously found that on the basis of other measurements the extremes are Lycia on the one hand and the Sphakiots and Selinots on the other. It would appear therefore that some special factor was at work differing from those that we have previously met with.

It has been suggested by Pearson and Crewdson Benington[29] that the personal equation in measurement may be responsible for this divergence. Until Hawes'

series are fully worked out or other data are available from elsewhere this is difficult to test on the series under examination. For small series the difference between v. Luschan's measurements on 49 men and Hawes' Sphakiots and Selinots (98 and 33 men respectively) is not great. The latter are worked out on *absolute measurements* and we give below v. Luschan's absolute measurements index. Comparing the absolute measurements Hawes' and v. Luschan's means of upper facial height agree to within a millimetre and the divergencies occur in the bizygomatic width, but even here the difference—under 3 mm.—is not too great to be the result of chance. The figures are:

Place	Authority	Bizygomatic Width	Upper Facial Height	Upper Facial Index
Sphakia	Hawes ...	140·6	76·7	54·56
„	v. Luschan	143	76	54·39
Selinos	Hawes ...	140·9	74·9	53·20
„	v. Luschan	143	75	52·51

Pearson's criticisms are based on skull measurements and are more than borne out by Martin's table[30] where the extreme range for the human race is given—by groups—as 49·0—56·0, but his remarks apply with still greater force to measurements on the living.

Summarizing them there are two reasons for rejecting the upper facial index as a racial test, first the possibility of error in measurements and secondly the small variation occurring in the human race. In regard to the first we have shewn that as far as can be judged from the limited data at our disposal independent observers have obtained similar results in Crete. In regard to the second the differences obtained in our area are sufficient to warrant a careful inquiry into the causes of this difference.

We still have a marked distinction between our two big groups Crete and Cyprus, but the conditions of the smaller groups are different. Crete is in an intermediate position, a group of nearly 100 men from the Peloponnese—sufficient at any rate to point the way to the true condition in Mani—at one end and Cyprus at the other.

Nasal Indices.

The nasal indices (see Table IX) all shew as is usual a high degree of variation. They fall into two groups, the Meligalots, Maniots, Selinots and Sphakiots on the one hand and Cypriots on the other. The third Cyprian subgroup (Enkomi area) has a low index, the Leukadians occupy an intermediate position, the remainder are broader nosed. It would appear that the narrow-nosed groups are rather more homogeneous than the remainder, but the difference between the standard deviations, except in the case of Cyprus, is not sufficient to be of any significance. The fact that the means for Crete and Cyprus, each island taken as a single unit, are almost in exact agreement is striking, but the difference between the standard deviations

(1·28 ± ·24) must be considered before any conclusion can be adopted. Finally it should be noted that the standard deviations in Cyprus are higher than elsewhere.

TABLE IX.

Nasal Indices.

Place	Number	Mean	Standard Deviation	Coefficient of Variation
Albania	133	67·36 ± ·44	7·53 ± ·31	11·18 ± ·47
Leukas	42	67·07 ± ·75	7·22 ± ·53	10·76 ± ·80
Meligala	33	65·95 ± ·66	5·64 ± ·47	8·55 ± ·75
Mani	99	62·36 ± ·41	6·02 ± ·29	9·65 ± ·48
Crete, Whole island	320	68·42 ± ·26	6·87 ± ·18	10·44 ± ·28
„ Selinos and Sphakia ...	51	65·76 ± ·51	5·45 ± ·36	8·29 ± ·56
Cyprus, Nicosia	111	69·59 ± ·55	8·66 ± ·39	12·45 ± ·56
„ Lapithos	221	69·47 ± ·36	7·94 ± ·25	11·42 ± ·37
„ Enkomi	167	64·25 ± ·36	6·91 ± ·25	10·75 ± ·40
„ Levkoniko	87	69·25 ± ·55	7·50 ± ·38	10·82 ± ·55
„ 4 groups combined ...	586	68·08 ± ·23	8·15 ± ·16	11·98 ± ·24

Stature.

There is little evidence of the stature in ancient Greece. v. Luschan's estimate of the Minoan stature is 1550 mm., 1600 mm. and 1650 mm. for three individuals; this is based on a humerus and two femora, neither the sex nor the method employed is stated. Duckworth's estimate for the Middle Minoan from Palaikastro is 1625.

Using Pearson's method of estimating stature for the Bronze Age stature in Cyprus, four females, two of which were reconstructed on the evidence of four long bones, one on the femur and humerus and one on the femur and tibia, had a mean stature of 1505 and one male 1602. The evidence is too slender to be of great value.

TABLE X.

Stature. Living Adult Males.

Place	Number	Mean	Standard Deviation	Coefficient of Variation
Crete, Whole island	318	170·61 ± ·26	6·75 ± ·18	3·96 ± ·12
„ Eparchies of Selinos and Sphakia only	50	175·26 ± ·54	5·71 ± ·39	3·26 ± ·22
Albanians	140	169·32 ± ·37	6·57 ± ·26	3·88 ± ·16
Cyprus, Whole island	585	168·77 ± ·17	6·16 ± ·12	3·64 ± ·07
„ Nicosia and neighbourhood ...	—	167·88 ± ·39	6·05 ± ·27	3·60 ± ·16
„ Lapithos	221	168·00 ± ·25	5·47 ± ·18	3·25 ± ·10
„ Enkomi	167	169·05 ± ·32	6·08 ± ·22	3·59 ± ·13
„ Levkoniko	87	168·98 ± ·46	6·37 ± ·33	3·77 ± ·19
„ Leukas	42	166·80 ± ·67	6·43 ± ·47	3·86 ± ·33
Lycian Gypsies	53	166·02 ± ·44	4·78 ± ·31	2·88 ± ·20
Alexandria	643	166·62 ± ·16	5·97 ± ·11	3·59 ± ·07
Giza	326	167·80 ± ·22	5·88 ± ·16	3·50 ± ·09

The living stature is available from Crete and Cyprus only in large numbers (see Table X); in both cases the stature is practically identical, 1690 and 1688 respectively. Three other small series are available all of which fall into a single shorter group, Leukas (1668), Mount Parnon (1662) and Lycians (1660).

The modern stature seems to be slightly greater than the ancient. In Cyprus this can be tested; two of the females were below the average in the measurements of the cranium, a third although she appeared to have been an adult when she died was certainly young, the fourth and shortest female had a head of average size, the male had the longest skull measured and a head breadth of average size. It may fairly be argued that the females were below the average stature and the male might be expected to be about average stature. Three of the females however came from one tomb, and one male and one female from a second. They therefore do not represent an average selection from the population.

Among the modern peoples we find first an unusual difference between the men from Selinos and Sphakia compared with the rest of the population of Crete. This difference is greater than any other difference in our series but until Hawes' figures are published cannot be satisfactorily explained. The differences between the various groups in Cyprus are small and do not give a clue to possible wide divergencies locally. The Cretans are however noticeably taller than the rest of the series. The significant differences may be tabulated.

TABLE XI.

Differences in Stature. Living Adult Males.

People compared		Difference
Leukas	Cyprus	1·97 ± ·69
,,	Crete	3·81 ± ·72
Crete	Cyprus	1·84 ± ·31

The figures suggest what had previously been deduced from the cephalic index, namely that in Crete we have an extreme type, and among the Lycian gypsies another extreme type, Leukas being more allied to the latter than to any of the other intermediates. We have again a general similarity in variation except among the Lycian gypsies who have a lower standard deviation than any of the other peoples.

PIGMENTATION.

Exact information in regard to hair and eye colour among the Greeks is scanty. Pittard remarks that brown eyes are in the majority, grey eyes frequently occur and blue are not rare. Ornstein* out of a total of 1767 men found 170 with blond hair, 1561 with brown hair and 36 with black. Weisbach's figures are very different: out of 47 cases 20 had black hair, two fair hair and the remainder various shades of brown; two out of the 47 had blue eyes. Schiff believed that he

* *Verhandlungen der Berliner anthrop. Gesellschaft*, 1877, S. 39 u. 1879, S. 305.

could make a distinction between the Southern and Northern Maniots. If our figures are correct we get three groups, (1) Albania and Cyprus, (2) Meligala and Mani, (3) Crete. In order to make the figures comparable they have been reduced to percentages removing all white haired people.

Hair Colour Percentages.

	Albania	Cyprus	Crete	Meligala	Mani
Fair ...	5·1	3·20	10	15	12
Brown ...	54·0	56·16	17	36	30·5
Black ...	40·9	40·64	74	48	57·5

A point which deserves special attention is that although Cyprus and Albania contain the least amount of dark hair they also contain the least fair hair; the Cretans, although the greater proportion of them have dark hair (Braun-schwarz and Schwarz), do not contain the least amount of fair hair. The personal equation of the observers should perhaps be considered responsible for the divergency but the Albanian observer also belonged to v. Luschan's school as well as the observers for Crete and the Peloponnese and Messenia.

Eye Colour Percentages.

	Albania	Cyprus	Crete	Meligala	Mani
Blue	10·20	9·58	5·3	9·1	6
Intermediate ...	48·20	39·02	25·4	33·3	36·5
Brown	41·60	51·40	69·3	57·6	57·5

Taking the eye colour we get rather a similar grouping, Albania and Cyprus being the most clearly allied and the Cretans who had the largest number of persons with dark hair have the smallest proportion of blue eyes.

There are two points of special importance to which attention should be drawn. First the number of blue-eyed persons occurring in some parts of the Greek world. In Albania and Cyprus about one man in ten has blue eyes and even in dark Crete one man in twenty. To take a practical example one would meet a blue-eyed man in Crete or the Peloponnese rather more frequently than one would meet a red-headed man in Oxford. The fact however that there appears to be such a difference in the distribution of blue eyes would possibly account for the blue-eyed tribes of classical times.

The second point of importance is that pigmentation does clearly divide up the two branches of the Alpine peoples. Among v. Luschan's Lycian gypsies every person examined has brown eyes and dark hair. How far this is a correct estimate it is difficult to decide as it is not quite in agreement with Twarjanowitsch's results from Armenia where he finds a population with a cephalic index of 86·89, and with 70 % having the eyes brown and the hair black—a close resemblance to Crete!

It is extremely difficult then to come to any conclusion about the pigmentation of the Greeks, but as far as our evidence goes at present we can affirm that blue eyes locally are not uncommon and that pigmentation does not appear to bear a definite correlation to cephalic index.

Our figures as a whole appear to shew that we have in the Greek world considerably different local types which conform as far as our evidence goes at present more to a short brachycephalic race, than to any other type; it is possible to form certain groups which on the whole with such limited evidence as is at our disposal conform, the Cretans and especially v. Luschan's Selinots and Sphakiots being at one end of the scale, i.e. tallest and most long-headed, the Lycian gypsies at the other. We are not inclined to lay much stress on the grouping which results from a study of the means alone: taken in conjunction with the question of variation it would appear to be of great importance.

If then we compare the variations as a whole, certain points of importance appear (see Table XII in which the standard deviations of indices and the

TABLE XII.

Provenance	σ			V		
	Cephalic Index	Upper Facial Index	Nasal Index	Stature	Glabello-Occipital Length	Head Breadth
Albania	4·44 ± ·17	3·20 ± ·13	7·53 ± ·31	3·85 ± ·18	—	—
Leukas	3·86 ± ·28	3·09 ± ·21	7·22 ± ·53	3·86 ± ·33	3·51 ± ·29	3·44 ± ·28
Meligala	3·21 ± ·26	4·02 ± ·33	5·64 ± ·47	—	2·63 ± ·23	3·75 ± ·31
Mani	3·77 ± ·18	3·73 ± ·18	6·02 ± ·29	—	3·11 ± ·23	3·40 ± ·18
Crete (Pyrgiótissa) ...	3·89 ± ·23	—	—	--	3·71 ± ·25	4·09 ± ·25
„ (Selinos and Sphakia)	3·91 ± ·26	2·95 ± ·20	5·45 ± ·36	2·95 ± ·22	3·98 ± ·31	3·27 ± ·22
„ Whole island ...	4·24 ± ·11	—	6·87 ± ·18	3·96 ± ·12	3·76 ± ·11	4·00 ± ·11
Lycian Gypsies	2·83 ± ·19	2·59 ± ·17	—	2·88 ± ·20	3·25 ± ·23	2·73 ± ·18
Cyprus, Group I	3·99 ± ·18	3·92 ± ·18	8·66 ± ·39	3·60 ± ·16	3·38 ± ·15	3·38 ± ·15
„ „ II	4·02 ± ·13	2·69 ± ·09	7·94 ± ·25	3·25 ± ·10	3·78 ± ·12	3·48 ± ·11
„ „ III	4·63 ± ·17	3·61 ± ·13	6·91 ± ·25	3·59 ± ·13	3·60 ± ·13	3·63 ± ·13
„ „ IV	4·25 ± ·22	3·87 ± ·20	7·50 ± ·38	3·77 ± ·19	3·65 ± ·19	3·49 ± ·18
„ Groups combined	4·09 ± ·08	3·71 ± ·07	8·15 ± ·16	3·64 ± ·07	3·87 ± ·08	3·59 ± ·07

coefficients of variation of absolute measurements have been tabulated). Unfortunately few measurements have been published on Lycian Greeks and Turks and they cannot therefore be included in our Tables. There is a general agreement in variation but the Cretans appear to be slightly more variable than the rest, this variation dating—on the evidence of cranial indices alone—at least from Middle Minoan times. On the other hand the series from Selinos and Sphakia measured by v. Luschan shew a remarkably low variation; we have shewn however that in some measurements Hawes obtained very different means, although he has not published his figures so that comparison of variations is impossible. We are inclined to believe therefore that v. Luschan did not obtain a normal sample of the population. It is not possible to discover whether his sample was due to some

artificial selection—his men are tall and may have been picked for this quality—or whether in those eparchies there does exist an unmixed strain.

The Lycian gypsies represent a consistently homogeneous strain with low variations in every case except in the glabello-occipital length where they have a normal coefficient of variation (i.e. one similar to the rest). In this latter case we have a curious exception in the low variation occurring among the people of Meligala, but no stress should be laid on this owing to the very small sample.

If we admit the Lycian gypsies to be a homogeneous people, and take into consideration their social *milieu*, it becomes of importance to find if possible another people who practised endogamy for similar religious reasons. The communities of the Druses and Maronites present an exact parallel, but no measurements appear to be available for statistical purposes. There exist however a number of anthropological measurements of Jews. Table XIII summarizes some of these. It will be seen that with very few exceptions they present features which are remarkably

TABLE XIII.

Measurements of Jews.

Provenance	No.	Cephalic Index		Nasal Index		Stature	
		M	σ	*M*	σ	*M*	*V*
Askenazim	202	83·7 ± ·14	3·0 ± ·10	—	—	—	—
Shephardim	149	81·5 ± ·18	3·2 ± ·12	—	—	—	—
Grusinians	71	86·7 ± ·34	4·2 ± ·24	—	—	—	—
Yemen	57	79·0 ± ·29	3·2 ± ·20	—	—	—	—
Aleppo	43	84·5 ± ·32	3·1 ± ·23	—	—	—	—
Morocco	30	80·2 ± ·28	2·3 ± ·20	—	—	—	—
Various Countries ...	500	81·42 ± ·10	3·16 ± ·07	—	—	—	—
Italy	112	82·45 ± ·25	3·87 ± ·17	—	—	—	—
Russian	139	81·09 ± ·20	3·58 ± ·14	—	—	—	—
European (? Germany)	100	80·3 ± ·21	3·14 ± ·15	—	—	—	—
South Russia ...	100	81·94 ± ·17	2·48 ± ·12	—	—	—	—
Persian Jews ...	57	81·32 ± ·35	3·92 ± ·25	60·37 ± ·56	6·07 ± ·40	164·35 ± ·52	3·53 ± ·22
Yemen	78	74·49 ± ·25	3·27 ± ·18	57·41 ± ·42 (50)	4·35 ± ·29	159·40 ± ·46	3·76 ± ·20
Samarkand	100	83·69 ± ·28	4·09 ± ·19	58·66 ± ·55 (50)	5·67 ± ·38	166·42 ± ·39	3·52 ± ·17

consistent. The data have been collected by different observers, and the countries represented are world wide. Standard deviations below 3 in the cephalic index occur in two cases only, in one of which however so small a number was measured that the figure is uncertain, the other case from South Russia, in which 100 individuals are included, is noticeable; all the other cases in the table fall within the limits of 3—4, except 71 Grusinians whose standard deviation is 4·2 and the Jews from Samarkand (σ 4·09). It is impossible to discuss at length here the vexed question of the Jewish race, but it would appear that we have in the above table a consistent degree of variation in a race which has on the whole been endogamous although exposed to many vicissitudes of climatic conditions, suggesting that such

hybridisation as has occurred is not recent but of very early date; their original centre of dispersion is the Eastern shore of the Mediterranean and it would appear that their racial origin is similar to that which we shall shew later appears probable for the other inhabitants of the Eastern Mediterranean.

In attempting to decide the racial position of the people under discussion we are clearly faced with the difficulty that on general grounds we cannot place a greater (or a less) reliance on the cephalic index than on other features. Our figures however shew that of all the material available the cephalic index is less variable and therefore should prove a good guide even if it does not possess the ideal attributes assigned to it by Ripley. Moreover, as we have already shewn, most of the other characters would lead us to establish a grouping not unlike the grouping on which we should have to depend if the cephalic index were our only guide. Clearly then the group-complex of what may be called "racial characters" expresses a definite entity by which one group may be distinguished from another. We are however met by the great difficulty that these groups are in our area by no means equally dispersed although this is possibly due to the incomplete nature of our data. If we adopt such a method of classification we cannot admit any of the inhabitants of the Eastern Mediterranean to homogeneity with the "Mediterranean Race" as exemplified in Corsica, Sardinia and Egypt, but must consider them as a mixed race.

The only clear representatives of the "Armenoid" race are the Bektasch and Tadchadsky of Lycia, but the Leukadians have certain points in common with them. We have a series of groups which do not appear to be homogeneous occupying an intermediate position. There are some remarkable factors which need attention. In the homogeneous groups and even among the heterogeneous groups we may go great distances and meet little variation. It is a far cry from Giza to Corsica, yet the cephalic index of two series from these two groups is closely akin; again from Lycia to the Peloponnese is within our area a comparatively long journey, yet the resemblance between the Lycians and the Maniots is in some respects most striking. On the other hand if we cross the mountains and travel from the Bay of Salamis in Cyprus to the North coast—a leisurely day's journey—we find that the two populations are dissimilar. We have shewn however that there are two racial types of comparative homogeneity at either end of the scale and that those peoples who present these local divergencies are very variable, having in every case a high standard deviation. As far as our present evidence goes then the subdivision into numerous local types would appear to serve no useful purpose. When we find two sets of peoples whose cephalic index differs by as much as 10 units but who appear to be comparatively homogeneous we are justified in considering them to belong to different groups, but when we find numerous smaller populations differing no doubt from the limiting groups and from each other but all possessing considerable heterogeneity we can most easily explain this heterogeneity by admitting racial admixture. This admixture does not appear to have been similar in every case, although in a few it has been. We have

not at present sufficient evidence to speak of Greece proper, but such evidence as we have would appear to justify the assertion that the numerous small communities of the ancient Mediterranean differed physically; thus we may say that there was a physical background to the struggles between Amathus and Salamis, and probably between the cities of Greece. To suppose that it is possible to establish a Greek type and to distinguish between Hellene and Barbarian φύσει does not appear justifiable.

It has been also suggested that there are two "types" in Greece—a brachycephalic Alpine and a long-headed Mediterranean—and that these can be distinguished. Now we have shewn that we do actually find such populations in a comparatively pure state in certain parts of the Mediterranean. That they exist in Greece is certain and it is not difficult to pick out of a large number of examples of both types. To say however that the population is made up of two types is unsatisfactory, because though it is possible and indeed probable that both types have participated in forming the present population, the latter seems at present to be so inextricably mixed that its separate elements are no longer distinguishable.

It may be argued that a large degree of variability is not due to admixture but is rather an attribute of the race. At present the series of crania of ancient date are so rare that no statistical evidence of variation in ancient times is available, it would appear however a more reasonable supposition to admit racial admixture when at either end of the scale it is possible to select specimens both of the Mediterranean and Alpine type, especially when we find a race that is admittedly pure has a low degree of variation.

It has been held by some classical scholars that the Nordic race has contributed to the population of the Eastern Mediterranean. It has been felt necessary to account for elements in the population who had fair hair and light eyes. It must however be remembered that colour terms are necessarily relative and among a population so dark as the Mediterranean peoples the standard is necessarily different from the standard of Northern Europe. A small proportion of blue-eyed persons does still occur in the Eastern Mediterranean and is certainly commoner in some areas than others. In Cyprus the blue-eyed persons appeared as far as a careful examination could determine to be akin to the "Armenoid" race and no trace of Nordic types was found. Individuals, or groups with a number of individuals, possessing features of this type, would sufficiently account for the frequent references in classical writers to fair peoples, without presuming Nordic blood.

We have so far limited the discussion to race in the older sense, that is apart from environment. It is possible that environmental conditions have played a certain part in determining the numerous local variations which are to be met with in the area we are examining, but we have not at present sufficient evidence upon which to generalize.

In regard to the historical ethnology again evidence is lacking, people akin to the Mediterranean race were inhabiting Egypt in predynastic times. v. Luschan's argument (quoted on p. 93 above) that the earliest inhabitants of Lycia were

Armenoids can be combated by the equally plausible, but perhaps also invalid, argument that these curious endogamous religious communities are the remains of later invaders who refused to ally themselves with the people of the land. The mixture of population had taken place in Cyprus—or among the inhabitants of Cyprus in an earlier home—before the beginning of the Bronze Age and it appears to have been taking place in Crete at about the same time. Elsewhere evidence is entirely lacking.

It is impossible on physical grounds at present to come to any conclusions in regard to the nature of the Pelasgians*. In regard to the Achaeans we have shewn that there appears to be no good ground for suspecting the presence of Nordics.

Ripley (p. 410) speaking of the classical Greeks says: "The facial features seem to be well demonstrated in the classic statuary although it is curious as Stephanos observes that these ideal heads are distinctly brachycephalic. Either the ancient sculptors knew little of anthropology or else we have again a confirmation of our assertion that however conscious of their peculiar facial traits a people may be, the head form is a characteristic whose significance is rarely recognized." We have attempted to shew that brachycephaly certainly did occur among the Greeks of the classical period. The small series of crania at our disposal contain both long-headed and short-headed types and we cannot agree with Ripley's dictum that the primitive Hellenic type was of pure Mediterranean stock. It is possible that the earliest dwellers in Greece may have been Mediterranean people—evidence is lacking on that point—but long before Hellenic civilization had developed the inhabitants of the Greek world appear, as far as our scanty data go, to have been similar generally to the inhabitants of that area today.

Their further history we cannot exactly follow. We know of movements of peoples, of irruption of Slavs, and of Turks. It is not improbable that these late comers were physically akin to the people who were already in occupation. With the Turk and possibly before came a large admixture of Negroid elements. Exactly how far the population has been affected by this influence it is difficult to say, but it forms an interesting example of a definite introduction of alien blood in the population, which has up to the present not been absorbed.

In smaller quantities and at various times, notably under the Byzantine Empire and during the Crusades, it is not unlikely that Nordic influence made itself felt in the Eastern Mediterranean but this influence cannot be detected at present.

While admitting then the presence of numerous minor differences, sufficiently great to make it necessary to know the exact provenance of any anthropological data we may wish to examine, it would not seem possible to assign any definite racial position to the greater part of the people of the Eastern Mediterranean but

* Cf. Treidler, "Alte Völker der Balkenhalbinsel," *Archiv für Anthropologie*, xi. S. 97: "In einem Punkte muss man Beloch recht geben die ganze thrakische Frage ist insofern belanglos als zwischen Thrakern und Hellenen in physischer Beziehung kein wesentlicher Unterschied bestanden hat." The same author continues: "no doubt the 'Urpelasger' were identical with the Thracians."

rather to class them as representing a combination probably of comparatively early date of Alpine and Mediterranean both of which stocks are found sporadically in a comparatively unmixed state in some parts of the area.

DESCRIPTION OF PLATE III.

Figs. 1, 2. Athenian cranium Fourth Century B.C.
Figs. 3, 4. Modern Greek (Cyprus).
Figs. 5, 6. Athenian cranium Third Century B.C.

The crania are in the University Museum, Oxford.

Measurements.

Fig. 1.	Fig. 2.
Naso-occipital length	Minimum Frontal Diameter
181	99·5
Fig. 5.	**Fig. 6.**
Naso-occipital length	Minimum Frontal Diameter
186	94

No measurements available for Figs. 3 and 4.

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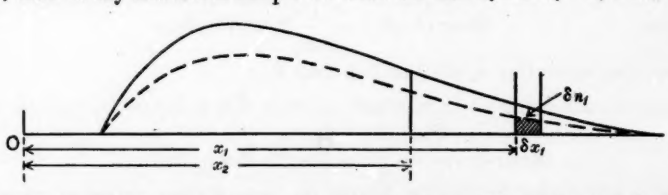
ON THE PROBABLE ERRORS OF FREQUENCY CONSTANTS.

PART III.

EDITORIAL.

(1) THE two previous parts* of this *résumé* of the theory of probable errors dealt with the probable errors of moments, and deduced the probable error of any frequency constant by considering that constant as a function of the moments, the probable errors of which had earlier been investigated. This third section proceeds from a different standpoint; it treats of the probable errors of constants supposed to be determined by a knowledge of the ranges in which certain proportions of the frequency lie. The fundamental papers on the subject are Edgeworth's "Exercises in the Calculation of Errors" (*Phil. Mag.* Vol. 36, pp. 98 *et seq.* 1893) and Sheppard's "On the Application of the Theory of Error to Cases of Normal Distribution and Correlation" (*Phil. Trans.* Vol. 192, pp. 130 *et seq.* 1898). We are unable to follow Edgeworth's reasoning, particularly that portion of it in the footnote on p. 99, where he states that the displacements of the two quartiles and the median are independent. They have, as we shall see, considerable correlation. With Dr Sheppard's numerical conclusions we are generally in agreement.

(2) We start from a population distributed according to any law of frequency, the ordinate at x_1 being y_1', and m_1 giving the total frequency beyond y_1' on the right, say. The total population M is supposed very large, and a sample N is taken from it, small as compared to M, but such that $1/\sqrt{N}$ is small as compared to unity. Let $\bar{n}_1 = m_1 \times N/M$ be the mean quantity that would be found beyond y_1' in many samples, then in any individual sample we shall not find $\bar{n}_1$ but $n_1 = \bar{n}_1 + \delta n_1$ beyond



y_1 or x_1 of the sampled population. Accordingly the ordinate which cuts off $\bar{n}_1/N$ of the sample frequency will not be at the point x_1 but at some distance δx_1 from it and the area of the dotted curve which represents the sample on δx_1 will be δn_1. Thus if $y_1 = y_1' N/M$ be the ordinate at x_1 of the sample we shall have to

* *Biometrika*, Vol. II. pp. 273—281 and Vol. IX. pp. 1—10. As in the earlier papers, this paper has been reproduced from Lecture Notes.

a first approximation, i.e. if we may neglect the terms in $\frac{1}{N}$ as compared with those in $\frac{1}{\sqrt{N}}$:

$$y_1 \delta x_1 = \delta n_1 \quad \ldots\ldots(1).$$

Similarly for a second ordinate

$$y_2 \delta x_2 = \delta n_2 \quad \ldots\ldots(1\ bis).$$

Such equations connect the change in the grade with the change in the frequency and provide at once the relations

$$\sigma_{x_1} = \sigma_{n_1}/y_1, \quad \sigma_{x_2} = \sigma_{n_2}/y_2 \quad \ldots\ldots(2).$$

If it be suggested that we shall have for a given sample a whole series of very irregular distributions of the n_1 individuals beyond y_1, the reply is that we are seeking to find σ_{x_1} for all samples, and we are justified to our degree of approximation in considering that δx_1 is the average grade change for *all* distributions which give an excess δn_1 beyond y_1, and that such average grade change may be looked upon as resulting from an area change and not from a scheme of isolated individuals.

Now if N be large as compared to M we have at once

$$\left.\begin{aligned} \sigma^2_{n_1} &= N \frac{m_1}{M}\left(1 - \frac{m_1}{M}\right) \\ \sigma^2_{n_2} &= N \frac{m_2}{M}\left(1 - \frac{m_2}{M}\right) \end{aligned}\right\} \quad \ldots\ldots(3),$$

which lead to

$$\left.\begin{aligned} \sigma^2_{x_1} &= \frac{N}{y_1^2}\frac{m_1}{M}\left(1 - \frac{m_1}{M}\right) = \frac{N}{y_1^2}\frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_1}{N}\right) \\ \sigma^2_{x_2} &= \frac{N}{y_2^2}\frac{m_2}{M}\left(1 - \frac{m_2}{M}\right) = \frac{N}{y_2^2}\frac{\bar{n}_2}{N}\left(1 - \frac{\bar{n}_2}{N}\right) \end{aligned}\right\} \quad \ldots\ldots(4).$$

To our order of approximation $\left(\text{terms in } \frac{1}{\sqrt{N}}\right)$ we write finally

$$\sigma_{x_1} = \frac{1}{\sqrt{N}}\left(\frac{N}{y_1}\right)\sqrt{\frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_1}{N}\right)}, \quad \sigma_{x_2} = \frac{1}{\sqrt{N}}\left(\frac{N}{y_2}\right)\sqrt{\frac{\bar{n}_2}{N}\left(1 - \frac{\bar{n}_2}{N}\right)} \quad \ldots\ldots(5).$$

Again $$\text{Mean}\,(\delta x_1 \delta x_2) = \frac{1}{y_1 y_2}\,\text{Mean}\,(\delta n_1 \delta n_2).$$

Now if we agree that x_2 shall be less than x_1,

$$n_2 = n_1 + n_2 - n_1 = n_1 + n_3 \text{ say},$$
$$\delta n_2 = \delta n_1 + \delta n_3,$$
$$\text{Mean}\,(\delta n_1 \delta n_2) = \text{Mean}\,(\delta n_1)^2 + \text{Mean}\,(\delta n_1 \delta n_3).$$

Now n_1 and n_3 are frequencies having no part of their ranges in common and accordingly if N be large as compared with M,

$$\text{Mean}\,(\delta n_1 \delta n_3) = -N\frac{m_1}{M}\frac{m_3}{M}$$
$$= -N\frac{\bar{n}_1}{N}\frac{\bar{n}_3}{N} \quad \ldots\ldots(6),$$

with the same approximation as before.

Accordingly

$$\text{Mean}\,(\delta n_1 \delta n_2) = N\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right) - N\frac{\bar{n}_1}{N}\left(\frac{\bar{n}_2-\bar{n}_1}{N}\right),$$

or

$$\text{Mean}\,(\delta n_1 \delta n_2) = N\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right) \quad \ldots\ldots\ldots\ldots(7).$$

Thus

$$\text{Mean}\,(\delta x_1 \delta x_2) = \frac{N}{y_1 y_2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right) \quad \ldots\ldots\ldots\ldots(8).$$

Equations (5) and (8) are the fundamental equations of our present subject. They give the standard deviations and correlations of the errors in any lengths measured along the x-axis as determined by the frequencies of the corresponding ranges.

If we have

$$z = f(x_1, x_2, \ldots x_p),$$

then to our degree of approximation,

$$\sigma_z^2 = S\left\{\left(\frac{df}{dx_s}\right)^2 \sigma^2_{x_s}\right\} + 2S'\left\{\left(\frac{df}{dx_s}\frac{df}{dx_{s'}}\right) \times \text{Mean}\,(dx_s dx_{s'})\right\} \quad \ldots\ldots\ldots(9),$$

where S denotes summation of s from 1 to p and S' summation of every pair of different s and s' from 1 to p.

With our degree of approximation, i.e. to first order terms in $1/\sqrt{N}$, the above results are true whatever be the law of frequency.

For the special case $z = x_2 - x_1$ we find

$$\sigma^2_{x_2-x_1} = \frac{1}{N}\left\{\frac{N^2}{y_1^2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right) + \frac{N^2}{y_2^2}\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_2}{N}\right) - 2\frac{N^2}{y_1 y_2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right)\right\}$$
$$\ldots\ldots(10).$$

For the special case that $x_2 - x_1 = l_{q_R q_L}$, the intraquartile range, $\frac{\bar{n}_1}{N} = \frac{1}{4}$, $\frac{\bar{n}_2}{N} = \frac{3}{4}$,

$$\sigma^2_{l_{q_R q_L}} = \frac{1}{N}\left\{\frac{3}{16}\left(\frac{N^2}{y^2_{q_R}} + \frac{N^2}{y^2_{q_L}}\right) - \frac{2}{16}\frac{N^2}{y_{q_R} y_{q_L}}\right\} \quad \ldots\ldots\ldots\ldots(11).$$

For a symmetrical distribution $y_R = y_L = y_q$ say, and

$$\sigma_{l_{qq}} = \frac{\sigma}{\sqrt{2N}}\frac{N}{\sqrt{2}\sigma y_q} \quad \ldots\ldots\ldots\ldots(11\,bis).$$

For the special case that $x_2 - x_1 = l_{mq_R}$, the median to right quartile range,

$$\sigma^2_{l_{mq_R}} = \frac{1}{N}\left(\frac{3}{16}\frac{N^2}{y^2_{q_R}} + \frac{1}{4}\frac{N^2}{y^2_m} - \frac{1}{4}\frac{N^2}{y_{q_R} y_m}\right) \quad \ldots\ldots\ldots\ldots(12),$$

and for $x_2 - x_1 = l_{q_L m}$, the left quartile to median range,

$$\sigma^2_{l_{q_L m}} = \frac{1}{N}\left(\frac{1}{4}\frac{N^2}{y^2_m} + \frac{3}{16}\frac{N^2}{y^2_{q_L}} - \frac{1}{4}\frac{N^2}{y_{q_L} y_m}\right) \quad \ldots\ldots\ldots\ldots(12\,bis).$$

(12) and (12 *bis*) will only give the same result provided $y_{q_R} = y_{q_L}$, which is of course satisfied by symmetrical curves but might be satisfied by other curves as well.

(3) We may now examine some of the resulting error correlations. If $r_{x_1x_2}$ be the correlation in errors of x_1 and x_2 we have by (8)

$$r_{x_1x_2} = \sqrt{\frac{\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right)}{\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_1}{N}\right)}} \quad \text{.....(13).}$$

For special cases:

(*a*) Quartile and Quartile: $r_{q_Lq_R} = \frac{1}{3} = \cdot 3333,$

(*b*) Median and Quartile: $r_{mq_L} = r_{mq_R} = \frac{1}{\sqrt{3}} = \cdot 5774,$

(*c*) Median and Decile: $r_{md_L} = r_{md_R} = \frac{1}{3\sqrt{3}} = \cdot 1925,$

(*d*) Decile and Decile: $r_{d_Ld_R} = \frac{1}{9} = \cdot 1111,$

(*e*) Quartile and Decile: $r_{d_Rq_R} = r_{d_Lq_L} = \frac{1}{\sqrt{3}} = \cdot 5774,$

(*f*) Quartile and Decile: $r_{d_Rq_L} = r_{d_Lq_R} = \frac{1}{3\sqrt{3}} = \cdot 1925,$

(*g*) Median and Quarto-decile: $r_{mqd_R} = r_{mqd_L} = \sqrt{\frac{2}{3}} = \cdot 8165.$

It is clearly impossible to treat such correlations as zero*. It is of interest to note that they are absolutely independent of the nature of the frequency distribution, and approach unity as the correlated grades approach each other.

Now if we consider the determination of a range from the mean and mean square aspect we shall have

$$x_1 = \bar{x} + \lambda_1\sigma, \quad x_2 = \bar{x} + \lambda_2\sigma,$$

where λ_1 and λ_2 are quantities to be determined from the form of the frequency distribution when we know its nature.

Thus $$\sigma^2_{x_1} = \sigma^2_{\bar{x}} + \lambda_1^2\sigma_\sigma^2 + 2\lambda_1 \times \text{Mean}\,(\delta\bar{x}\,\delta\sigma),$$

$$\sigma^2_{x_1-x_2} = (\lambda_1-\lambda_2)^2\sigma_\sigma^2.$$

Now $$\sigma^2_{\bar{x}} = \sigma^2/N, \quad \sigma_\sigma^2 = \frac{1}{4N}(\beta_2-1)\,\sigma^2$$

and $$\text{Mean}\,(\delta\bar{x}\,\delta\sigma) = \tfrac{1}{2}\sqrt{\beta_1}\,\sigma^2/N,$$

where β_1 and β_2 are the usual fundamental frequency constants. Accordingly

$$\sigma_{x_1} = \frac{\sigma}{\sqrt{N}}\left\{1 + \lambda_1\sqrt{\beta_1} + \tfrac{1}{4}\lambda_1^2(\beta_2-1)\right\}^{\frac{1}{2}} \quad \text{.....(14),}$$

$$\sigma_{x_1-x_2} = \frac{\sigma}{\sqrt{2N}}(\lambda_1-\lambda_2)\sqrt{\tfrac{1}{2}(\beta_2-1)} \quad \text{.....(15),}$$

$$\text{Mean}\,(\delta x_1\,\delta x_2) = \frac{\sigma^2}{N}\left\{1 + \tfrac{1}{2}(\lambda_1+\lambda_2)\sqrt{\beta_1} + \tfrac{1}{4}\lambda_1\lambda_2(\beta_2-1)\right\} \quad \text{.....(16).}$$

* See remark, p. 133.

(14), (15) and (16) give the magnitudes to be compared with (5), (10), and (8) respectively when results obtained by mean and standard deviation methods are to be compared for accuracy with those obtained by grade methods.

We may now consider what these results reduce to when the frequency follows the simple normal distribution.

We shall suppose

$$y_s = \frac{N}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x_s-\bar{x})^2}{\sigma^2}} \quad \text{.....(17),}$$

and write $h_s = (x_s - \bar{x})/\sigma$ and $H_s = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2}$,

thus $y_s = NH_s/\sigma$, and H_s may be found directly from the *Tables for Statisticians*, when h_s is known, and h_s will be known when $\bar{n}_s$ is known. Again, $\lambda_s\sigma = x_s - \bar{x}$, and therefore $\lambda_s = h_s$.

We obtain the following results:

For (5):

$$\sigma_{x_1} = \frac{\sigma}{\sqrt{N}} \frac{1}{H_1} \sqrt{\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right)}, \quad \sigma_{x_2} = \frac{\sigma}{\sqrt{N}} \frac{1}{H_2} \sqrt{\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_2}{N}\right)} \quad \text{.....(18).}$$

For (14):

$$\sigma_{x_1} = \frac{\sigma}{\sqrt{N}}(1+\tfrac{1}{2}h_1^2)^{\frac{1}{2}}, \quad \sigma_{x_2} = \frac{\sigma}{\sqrt{N}}(1+\tfrac{1}{2}h_2^2)^{\frac{1}{2}} \quad \text{.....(19).}$$

For (10):

$$\sigma_{x_1-x_2} = \frac{\sigma}{\sqrt{2N}} \left\{ \frac{2}{H_1^2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right) + \frac{2}{H_2^2}\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_2}{N}\right) - \frac{4}{H_1 H_2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right) \right\}^{\frac{1}{2}}$$

.....(20).

For (15)*:

$$\sigma_{x_1-x_2} = \frac{\sigma}{\sqrt{2N}}(h_1 - h_2) \quad \text{.....(21).}$$

For $r_{x_1x_2}$ from (13):

$$r_{x_1x_2} = \sqrt{\frac{\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right)}{\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_1}{N}\right)}} \quad \text{.....(22).}$$

For $r_{x_1x_2}$ from (18):

$$r_{x_1x_2} = \frac{1+\frac{1}{2}h_1h_2}{\sqrt{1+\frac{1}{2}h_1^2}\sqrt{1+\frac{1}{2}h_2^2}} \quad \text{.....(23)†.}$$

We may illustrate (23) first.

* The reader must bear in mind that with the present notation $h_1 > h_2$ and $n_1 < n_2$.

† h_1 and h_2 may be of opposite sign.

Correlation of errors of grades as found by mean and standard deviations.

(*a*) Quartile and Quartile: $r_{q_L q_R} = \cdot 6294$,

(*b*) Median and Quartile: $r_{m q_R} = \cdot 9026$,

(*c*) Median and Decile: $r_{m d_R} = \cdot 6152$,

(*d*) Decile and Decile: $r_{d_L d_R} = \cdot 0982$,

(*e*) Quartile and Decile: $r_{q_R d_R} = \cdot 9579$,

(*f*) Quartile and Decile: $r_{q_L d_R} = \cdot 3798$,

(*g*) Median and Quarto-decile: $r_{m q d_R} = \cdot 9694$.

It will be seen that the correlation of grade errors as thus deduced is far higher than when the grades are found directly from ranking.

We now turn to the position of the grade and investigate the S.D. of its error as found in the two different methods by (18) and (19):

TABLE I. *Accuracy as found from Ranking and from Moments.*

$\bar{n}$	Grade	Position from ranking	Position from Mean and S.D.	Ratio *
·9	1st Decile	$1\cdot7094\,\sigma/\sqrt{N}$	$1\cdot3495\,\sigma/\sqrt{N}$	1·2667
·8	2nd Decile	$1\cdot4288\,\sigma/\sqrt{N}$	$1\cdot1637\,\sigma/\sqrt{N}$	1·2278
·75	Left Quartile	$1\cdot3626\,\sigma/\sqrt{N}$	$1\cdot1079\,\sigma/\sqrt{N}$	1·2299
·7	3rd Decile	$1\cdot3180\,\sigma/\sqrt{N}$	$1\cdot0665\,\sigma/\sqrt{N}$	1·2358
·6	4th Decile	$1\cdot2680\,\sigma/\sqrt{N}$	$1\cdot0159\,\sigma/\sqrt{N}$	1·2482
·5	Median	$1\cdot2533\,\sigma/\sqrt{N}$	$1\cdot0000\,\sigma/\sqrt{N}$	1·2533
·4	6th Decile	$1\cdot2680\,\sigma/\sqrt{N}$	$1\cdot0159\,\sigma/\sqrt{N}$	1·2482
·3	7th Decile	$1\cdot3180\,\sigma/\sqrt{N}$	$1\cdot0665\,\sigma/\sqrt{N}$	1·2358
·25	Right Quartile	$1\cdot3626\,\sigma/\sqrt{N}$	$1\cdot1079\,\sigma/\sqrt{N}$	1·2299
·2	8th Decile	$1\cdot4288\,\sigma/\sqrt{N}$	$1\cdot1637\,\sigma/\sqrt{N}$	1·2278
·1	9th Decile	$1\cdot7094\,\sigma/\sqrt{N}$	$1\cdot3495\,\sigma/\sqrt{N}$	1·2667

It will be seen that increase of inaccuracy at all grades is singularly nearly constant lying between 23 °/₀ and 27 °/₀ and for rough purposes may be taken as 25 °/₀ throughout the usual range of grading in deciles.

We shall now work out two further series of values, namely $\sigma_{x_1 - x_2}$ as found (*a*) from left to right deciles or quartiles, and (*b*) from the median to these grades.

(*a*) Turning to (20) we put

$$H_2 = H_1, \text{ and } \frac{\bar{n}_2}{N} = 1 - \frac{\bar{n}_1}{N},$$

* The values in columns 3 and 4 were found to 6 decimals and the ratio of the S.D.'s found from these. Columns 3 and 4 must be multiplied by ·67449 if the probable error instead of the standard deviation be required.

and deduce, since $x_2 = -x_1$,

$$\sigma_{x_2 x_1} = \frac{\sigma}{\sqrt{2N}} \frac{2}{H_1} \sqrt{\frac{\bar{n}_1}{N}\left(1 - \frac{2\bar{n}_1}{N}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

TABLE II. *Accuracy of Determination of σ from Ranking symmetrically.*

$\bar{n}_1/N$	Range $2x_1$	σ_{2x_1}	σ_σ from σ_{2x_1}	Values from Mean and S.D.
$\frac{1}{40}$	$3{\cdot}92007\,\sigma$	$5{\cdot}2737\,\sigma/\sqrt{2N}$	$1{\cdot}3453\,\sigma/\sqrt{2N}$	The S.D. of the ranges are the coefficients of 2nd column $\times \frac{\sigma}{\sqrt{2N}}$. The ratios are therefore the coefficients of the 4th column. The S.D. of σ is of course $\sigma/\sqrt{2N}$, and the ratios are again coefficients of the 4th column
$\frac{1}{30}$	$3{\cdot}66787\,\sigma$	$4{\cdot}7514\,\sigma/\sqrt{2N}$	$1{\cdot}2954\,\sigma/\sqrt{2N}$	
$\frac{1}{20}$	$3{\cdot}28975\,\sigma$	$4{\cdot}1137\,\sigma/\sqrt{2N}$	$1{\cdot}2505\,\sigma/\sqrt{2N}$	
$\frac{1}{15}$	$3{\cdot}00219\,\sigma$	$3{\cdot}7178\,\sigma/\sqrt{2N}$	$1{\cdot}2384\,\sigma/\sqrt{2N}$	
$\frac{1}{14}$	$2{\cdot}93050\,\sigma$	$3{\cdot}6290\,\sigma/\sqrt{2N}$	$1{\cdot}2383\,\sigma/\sqrt{2N}$	
$\frac{1}{13}$	$2{\cdot}85219\,\sigma$	$3{\cdot}5358\,\sigma/\sqrt{2N}$	$1{\cdot}2397\,\sigma/\sqrt{2N}$	
$\frac{1}{12}$	$2{\cdot}76602\,\sigma$	$3{\cdot}4378\,\sigma/\sqrt{2N}$	$1{\cdot}2429\,\sigma/\sqrt{2N}$	
$\frac{1}{10}$	$2{\cdot}56310\,\sigma$	$3{\cdot}2233\,\sigma/\sqrt{2N}$	$1{\cdot}2576\,\sigma/\sqrt{2N}$	
$\frac{5}{10}$	$1{\cdot}68324\,\sigma$	$2{\cdot}4747\,\sigma/\sqrt{2N}$	$1{\cdot}4702\,\sigma/\sqrt{2N}$	
$\frac{1}{4}$	$1{\cdot}34898\,\sigma$	$2{\cdot}2252\,\sigma/\sqrt{2N}$	$1{\cdot}6495\,\sigma/\sqrt{2N}$	
$\frac{3}{10}$	$1{\cdot}04880\,\sigma$	$1{\cdot}9926\,\sigma/\sqrt{2N}$	$1{\cdot}8999\,\sigma/\sqrt{2N}$	
$\frac{4}{10}$	$\cdot50669\,\sigma$	$1{\cdot}4642\,\sigma/\sqrt{2N}$	$2{\cdot}8897\,\sigma/\sqrt{2N}$	

It will be clear from this table that the quartiles are not the best ranks from which to find the standard deviation with least error. The range should be that corresponding to $\frac{1}{14}$ to $\frac{1}{15}$ from each end of the series in the ranking, say $\frac{1}{14}$. This quatuordecimal range will provide σ with about 24 °/₀ greater probable error than the moment method. The quartiles give it with 65 °/₀ greater inaccuracy. The first and last deciles as a convenient divisor give only 26 °/₀ less accuracy. The use of the quartiles was undoubtedly adopted because of their relation in the normal curve to the probable error. But theoretically they are very inferior to the quatuordecimals.

(*b*) The table on the following page provides the data for the probable errors of σ when found from ranges whose terminals are not symmetrical.

It will be clear that the inaccuracy of the determination of σ from ranking increases with great rapidity as we cause our asymmetrical ranks to approach each other. Further the range taken on one side of the median gives a worse result than the same range placed symmetrically about the median. Or again, looked at in another way, the range corresponding to $\frac{4}{10}$ of the frequency on one side the median gives $\sigma_\sigma = 1{\cdot}93\sigma/\sqrt{2N}$, but the same frequency symmetrically placed about the median gives $\sigma_\sigma = 1{\cdot}47\sigma/\sqrt{2N}$. It might be supposed at first that our Table II could be derived from Table III by simply supposing that in doubling the frequency from which our determinations are made we have increased the accuracy

by $\sqrt{2}$. But the figures deduced in this way from the fifth column of Table III have for multipliers of $\sigma/\sqrt{2N}$

1·3660, 1·6036, 1·7887, 2·0406, 3·0150

instead of the multipliers

1·2576, 1·4702, 1·6495, 1·8999, 2·8897

TABLE III. *Accuracy of Determination of σ from Asymmetrical Ranks.*

$\bar{n}_2/N$	$\bar{n}_1/N$	Range $x_1 - x_2$	$\sigma_{x_1-x_2}$	σ_σ from $\sigma_{x_1-x_2}$	From Mean and S.D.
·5	·1	1·28155 σ	2·4757 $\sigma/\sqrt{2N}$	1·9318 $\sigma/\sqrt{2N}$	The S.D. of the ranges will be the coefficients of column 3 × $\sigma/\sqrt{2N}$. The S.D. of the S.D. will be throughout $\sigma/\sqrt{2N}$
·5	·2	·84162 σ	1·9087 $\sigma/\sqrt{2N}$	2·2678 $\sigma/\sqrt{2N}$	
·5	·25	·67449 σ	1·7062 $\sigma/\sqrt{2N}$	2·5296 $\sigma/\sqrt{2N}$	
·5	·3	·52440 σ	1·5134 $\sigma/\sqrt{2N}$	2·8859 $\sigma/\sqrt{2N}$	
·5	·4	·25335 σ	1·0803 $\sigma/\sqrt{2N}$	4·2639 $\sigma/\sqrt{2N}$	
·9	·3	1·80595 σ	2·7114 $\sigma/\sqrt{2N}$	1·5014 $\sigma/\sqrt{2N}$	
·8	·1	2·02317 σ	2·8808 $\sigma/\sqrt{2N}$	1·4239 $\sigma/\sqrt{2N}$	
·9	$\frac{1}{14}$	2·74680 σ	3·4327 $\sigma/\sqrt{2N}$	1·2497 $\sigma/\sqrt{2N}$	
·9	$\frac{1}{20}$	2·92643 σ	3·6974 $\sigma/\sqrt{2N}$	1·2634 $\sigma/\sqrt{2N}$	

of the last five entries of the fourth column of the table on p. 119. The explanation of this result is the relatively high correlation between x_1 and x_2 which reduces the higher values above resulting from a supposed independence of errors in the determination of our two quartile distances. Physically this signifies that if we were to determine one median to quartile distance from a first sample of N, and the other median to quartile distance from a second sample, we should get a less accurate result than determining both quartiles on the *same* sample. For the principle of correlation shows us that in the same sample if we get an excess in one quartile distance we shall on the average get a defect in the other. We thus may actually lose in accuracy when we combine measurements drawn from different samples in estimating the constants of a sampled population.

(4) The next question to be raised is whether the median can be obtained more accurately than by halving our total frequency. We have seen that the error of this process is $1{\cdot}2533\sigma/\sqrt{N}$. We might, however, obtain it as the half of any two grades on either side the median at equal distances from it, i.e.

$$m = \tfrac{1}{2}(x_1 + x_2).$$

Let each of the grades cut off $\bar{n}_1$ from the tails of the frequency, then

$$\begin{aligned}\sigma_m &= \tfrac{1}{2}\sigma_{x_1+x_2} \\ &= \frac{\sigma}{\sqrt{2N}}\left\{\frac{1}{H_1^2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right)+\frac{1}{H_1^2}\frac{\bar{n}_1}{N}\frac{\bar{n}_1}{N}\right\}^{\frac{1}{2}} \\ &= \frac{\sigma}{\sqrt{N}}\frac{1}{H_1}\sqrt{\frac{\bar{n}_1}{2N}} \qquad \text{.....(25)}\end{aligned}$$

To find when this is a maximum, we must differentiate with regard to $\bar{n}_1$ and put $d\sigma_m/d\bar{n}_1 = 0$ or

$$\frac{1}{2\bar{n}_1} = \frac{dH_1}{d\bar{n}_1} = \frac{1}{H_1}\frac{dH_1}{dh_1}\frac{dh_1}{d\bar{n}_1} = \frac{1}{N}\frac{h_1}{H_1}.$$

Thus

$$\frac{\bar{n}_1}{N} = \frac{H_1}{2h_1} \quad \text{.....(26).}$$

Putting

	$H_1/2h_1$	$\bar{n}_1/N$
$h_1 = \cdot60$	·277,687	·274,253
·61	·271,487	·270,931
·62	·265,471	·267,629
·63	·259,629	·264,347

we see that h_1 lies between ·61 and ·62 and thence reach by interpolation ·612,0014 as the required value of h_1. This leads to

$$\bar{n}_1/N = \cdot270{,}268,$$

or the right value of $\bar{n}_1$ is a little more than the quartile, or $x_1 - \bar{x}$, instead of being $\cdot67449\sigma$, $= \cdot61200\sigma$. This leads us to

$$\sigma_m = 1{\cdot}1112\sigma/\sqrt{N} \quad \text{.....(27),}$$

or the median found from the $\bar{n}_1/N = \cdot270268$ grades is more than twice as accurate as when found directly.

As these grades do not differ very much from $\bar{n}_1/N = \cdot25$ or the quartiles we note that the σ_m of the median found from the two quartiles is

$$\sigma_m = 1{\cdot}1126\sigma/\sqrt{N} \quad \text{.....(28).}$$

The difference in accuracy between (27) and (28) is too small to be of any importance for most statistical purposes, and accordingly the quartiles may be used to determine the median, and this with double the accuracy of direct investigation of the mid-individual. We can put into a table the results for finding the median from each pair of the following series:

TABLE IV. *Accuracy of Determination of Median from Pairs of Symmetrical Grades.*

$\bar{n}_1/N$	σ_m	Symbol for median
$\frac{1}{14}$	$1{\cdot}3858\,\sigma/\sqrt{N}$	m_7
$\frac{1}{10}$	$1{\cdot}2741\,\sigma/\sqrt{N}$	m_6
$\frac{2}{10}$	$1{\cdot}1295\,\sigma/\sqrt{N}$	m_5
·25	$1{\cdot}1126\,\sigma/\sqrt{N}$	m_4
max. ·270268	$1{\cdot}1112\,\sigma/\sqrt{N}$	m_3
$\frac{3}{10}$	$1{\cdot}1139\,\sigma/\sqrt{N}$	m_2
$\frac{4}{10}$	$1{\cdot}1576\,\sigma/\sqrt{N}$	m_1
$\frac{5}{10}$	$1{\cdot}2533\,\sigma/\sqrt{N}$	m_0

(5) The question now arises as to whether we could still further reduce the above inaccuracies by combining these results in pairs in the best proportions, say λ_s and $\lambda_{s'}$ where $\lambda_s + \lambda_{s'} = 1$. Let the corresponding median value be $m_{ss'}$, then

$$m_{ss'} = \lambda_s m_s + \lambda_{s'} m_{s'},$$

$$\sigma^2_{m_{ss'}} = \lambda^2_s \sigma^2_{m_s} + \lambda^2_{s'} \sigma^2_{m_{s'}} + 2\lambda_s \lambda_{s'} [\delta m_s \delta m_{s'}]^*,$$

and making this a minimum for varying λ_s, $\lambda_{s'}$ we find

$$\left.\begin{aligned} \lambda_s &= \frac{\sigma^2_{m_{s'}} - [\delta m_s \delta m_{s'}]}{\sigma^2_{m_s} + \sigma^2_{m_{s'}} - 2[\delta m_s \delta m_{s'}]} \\ \lambda_{s'} &= \frac{\sigma^2_{m_s} - [\delta m_s \delta m_{s'}]}{\sigma^2_{m_s} + \sigma^2_{m_{s'}} - 2[\delta m_s \delta m_{s'}]} \\ \sigma^2_{m_{ss'}} &= \frac{\sigma^2_{m_s} \sigma^2_{m_{s'}} - [\delta m_s \delta m_{s'}]^2}{\sigma^2_{m_s} + \sigma^2_{m_{s'}} - 2[\delta m_s \delta m_{s'}]} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

Table IV supplies the needful values of the σ_m's and what we require are the mean products

$$\begin{aligned} [\delta m_s \delta m_{s'}] &= \tfrac{1}{4}[(\delta x_1 + \delta x_1')(\delta x_2 + \delta x_2')] \\ &= \tfrac{1}{4}([\delta x_1 \delta x_2] + [\delta x_1 \delta x_2'] + [\delta x_1' \delta x_2] + [\delta x_1' \delta x_2']) \\ &= \frac{1}{4} \frac{\sigma^2}{N H_s H_{s'}} \left\{ \frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_2}{N}\right) + \frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_2'}{N}\right) + \frac{\bar{n}_2}{N}\left(1 - \frac{\bar{n}_1'}{N}\right) + \frac{\bar{n}_2'}{N}\left(1 - \frac{\bar{n}_1'}{N}\right) \right\}. \end{aligned}$$

But $\bar{n}_1 = N - \bar{n}_1'$, $\bar{n}_2 = N - \bar{n}_2'$, thus finally

$$[\delta m_s \delta m_{s'}] = \frac{\sigma^2}{N} \frac{1}{H_s H_{s'}} \frac{\bar{n}_1}{2N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

where $\bar{n}_1$ corresponds to x_1 the outermost of the paired grades.

This leads us at once to

$$[\delta m_s \delta m_{s'}] = \sigma^2_{m_s} H_s / H_{s'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

which much simplifies the calculations, if we remember that σ_{m_s} and H_s correspond to the grade further from the median.

Hence for correlation of m_s, $m_{s'}$ we have

$$r_{m_s m_{s'}} = \frac{\sigma_{m_s} H_s}{\sigma_{m_{s'}} H_{s'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

Tables V and VI give first the mean product-moment $[\delta m_s \delta m_{s'}]$ and secondly the standard deviation $\sigma_{m_{ss'}}$ and the coefficients λ_s, $\lambda_{s'}$ of $m_{ss'} = \lambda_s m_s + \lambda_{s'} m_{s'}$.

It is clear from Table VI that the lowest value of the probable error of the median will be obtained by deducing the median from the two 1st deciles and from the two 3rd deciles and taking ·34388 of the first plus ·65612 of the second value; the value of $\sigma_{m_{ss'}}$ will then be $1{\cdot}0468\sigma/\sqrt{N}$, i.e. only about 5 °/₀ more inaccurate than that found by the arithmetic mean. We may search about this value for

* It is convenient to write $[\delta m_s \delta m_{s'}]$ to signify mean value of the product $\delta m_s \delta m_{s'}$.

TABLE V. *Values of* $[\delta m_s \delta m_{s'}] \Big/ \frac{\sigma^2}{N}$.

s \ s'	$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	·25	·270268	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{2}$
$\frac{1}{14}$	1·920,540	1·492,315	·935,478	·824,158	·791,690	·753,246	·677,891	·656,482
$\frac{1}{10}$	—	1·623,401	1·017,657	·896,557	·861,238	·819,416	·737,441	·714,152
$\frac{2}{10}$	—	—	1·275,856	1·123,943	1·079,666	1·027,238	·924,472	·895,274
·25	—	—	—	1·237,845	1·189,081	1·131,339	1·018,159	·986,004
·270268	—	—	—	—	1·234,835	1·174,871	1·057,036	1·023,944
$\frac{3}{10}$	—	—	—	—	—	1·240,793	1·116,663	1·081,398
$\frac{4}{10}$	—	—	—	—	—	—	1·339,936	1·297,619
$\frac{1}{2}$	—	—	—	—	—	—	—	1·570,796

TABLE VI. *Values of* $\sigma_{m_{ss'}} \Big/ \frac{\sigma}{\sqrt{N}}$ *and* λ_s, $\lambda_{s'}$.

s \ s'		$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	·25	·270268	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{2}$
$\frac{1}{14}$		1·3858	1·2620	1·0902	1·0604	1·0535	1·0475	1·0535	1·0895
	λ_s	—	·23437	·25680	·27395	·28190	·29462	·34759	·41972
	$\lambda_{s'}$	—	·76563	·74320	·72605	·71810	·70538	·65241	·58018
$\frac{1}{10}$		—	1·2741	1·0948	1·0624	1·0545	1·0468	1·0469	1·0748
	λ_s	—	—	·29886	·31952	·32894	·34388	·40478	·48511
	$\lambda_{s'}$	—	—	·70114	·68048	·67106	·65612	·59522	·51489
$\frac{2}{10}$		—	—	1·1295	1·0904	1·0800	1·0687	1·0559	1·0671
	λ_s	—	—	—	·42850	·44162	·46207	·54178	·63964
	$\lambda_{s'}$	—	—	—	·57150	·55838	·53793	·45822	·36036
·25		—	—	—	1·1126	1·0965	1·0809	1·0718	1·0780
	λ_s	—	—	—	—	·48408	·48683	·59427	·69898
	$\lambda_{s'}$	—	—	—	—	·51592	·51317	·40573	·30102
·270268		—	—	—	—	1·1112	1·0983	1·0800	1·0845
	λ_s	—	—	—	—	—	·52367	·61422	·72169
	$\lambda_{s'}$	—	—	—	—	—	·47633	·38578	·27831
$\frac{3}{10}$		—	—	—	—	—	1·1139	1·0938	1·0962
	λ_s	—	—	—	—	—	—	·64269	·75432
	$\lambda_{s'}$	—	—	—	—	—	—	·35731	·24568
$\frac{4}{10}$		—	—	—	—	—	—	1·1576	1·1551
	λ_s	—	—	—	—	—	—	—	·86587
	$\lambda_{s'}$	—	—	—	—	—	—	—	·13413
$\frac{1}{2}$		—	—	—	—	—	—	—	1·2533
	λ_s	—	—	—	—	—	—	—	—
	$\lambda_{s'}$	—	—	—	—	—	—	—	—

a lower grade of inaccuracy, but there is little advantage to be gained. Thus for $s' = \frac{1}{12}$, $s' = \frac{3}{10}$ we find

$$\sigma_{m_{ss'}} = 1{\cdot}0466\,\frac{\sigma}{\sqrt{N}}, \quad \lambda_s = {\cdot}31660, \quad \lambda_{s'} = {\cdot}68340 \quad \ldots\ldots\ldots\ldots(33),$$

only a very slightly less inaccuracy with a considerable change in the values of the λ's.

It is desirable in such a case to take round numbers easily remembered, and thus we may propose

$$m_{\frac{1}{10},\,\frac{3}{10}} = \tfrac{1}{3}m_{\frac{1}{10}} + \tfrac{2}{3}m_{\frac{3}{10}} \quad \ldots\ldots\ldots\ldots(34).$$

In this case

$$\sigma_{m_{\frac{1}{10},\,\frac{3}{10}}} = 1{\cdot}0469\,\frac{\sigma}{\sqrt{N}} \quad \ldots\ldots\ldots\ldots(35).$$

The practical difference of 1·0466 and 1·0469 is not worth considering and the coefficients are most easy to remember.

If we wish to better this at all substantially we must take three pairs of decile-determined medians. For example, suppose we take

$$m_{\frac{1}{10},\,\frac{3}{10},\,\frac{5}{10}} = \lambda_{\frac{1}{10}}m_{\frac{1}{10}} + \lambda_{\frac{3}{10}}m_{\frac{3}{10}} + \lambda_{\frac{5}{10}}m_{\frac{5}{10}}.$$

We find

$$\sigma_{m_{\frac{1}{10},\,\frac{3}{10},\,\frac{5}{10}}} = 1{\cdot}0321\,\frac{\sigma}{\sqrt{N}},$$

where

$$\left.\begin{aligned}\lambda_{\frac{1}{10}} &= {\cdot}33426\\ \lambda_{\frac{3}{10}} &= {\cdot}44795\\ \lambda_{\frac{5}{10}} &= {\cdot}21779\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(36).$$

Or very approximately,

$$\lambda_{\frac{1}{10}} = \tfrac{3}{9}, \quad \lambda_{\frac{3}{10}} = \tfrac{4}{9}, \quad \lambda_{\frac{5}{10}} = \tfrac{2}{9} \quad \ldots\ldots\ldots\ldots(37),$$

which give also to four decimals

$$\sigma_m = 1{\cdot}0321\,\frac{\sigma}{\sqrt{N}} \quad \ldots\ldots\ldots\ldots(38).$$

Whether this 2 °/₀ gain on using three instead of two median values is worth the extra labour, which is not, however, very great, may be discussable. Probably with the use of four to five medians we could make the accuracy substantially that of an arithmetic mean. The difficulties lie not so much in the reduction of the inaccuracy as in two other matters: (i) the whole superstructure of these probable errors of decile medians is based on the frequency distribution approaching that of the normal curve. We shall get in many cases larger errors than those of random sampling, due to definite deviations from such normality in our frequency; and (ii) the decile values, if there has been grouping in classifying the frequency, can only be determined by use of the probability integral tables. We cannot pick out the character of the individual who stands n_1 individuals from the top. He will lie somewhere in a group of p individuals, and unless these individuals be arranged in order of character, we cannot find the character of the n_1th and $(n_1 + 1)$th individual. Their characters would have to be found from those of the individuals who bound

their group by using the probability integral and actually determining from them the mean and standard deviation of the total frequency, the very quantities we are seeking to find by a more accurate process.

It may be said why not arrange the material in order of magnitude? The answer is that this is not an easy task if the observations run to hundreds or thousands. And further, while a mean may give accurately a result to $\frac{1}{100}$th of an inch, it may be far from desirable or possible to measure to more than $\frac{1}{2}''$, e.g. in the case of stature. Hence the decile individuals will be only known to $\frac{1}{2}''$ and the median found from a pair of deciles will only at best be known to the $\frac{1}{4}''$. The whole theory indeed supposes the characters of the decile individuals measured to very near the same accuracy as we need the mean. There are none of the advantages which arise from grouping after measuring in fairly coarse units, and then averaging up to get the mean. The application of the decile method should therefore be confined to cases where the numbers are not too large for ranking, where the distribution is approximately normal and where the character has been determined with considerable exactness at least for individuals in the region of the deciles. These limitations take much of our material out of the field of grade treatment.

(6) We can now adopt the same process as we have applied to the determination of the median to find if possible a more accurate value for the standard deviation by aid of ranking.

We take as before $$\sigma_{ss'} = \lambda_s \sigma_s + \lambda_{s'} \sigma_{s'},$$

where λ_s and $\lambda_{s'}$ are to be chosen so that

$$\sigma^2_{\sigma_{ss'}} = \lambda^2_s \sigma^2_{\sigma_s} + \lambda^2_{s'} \sigma^2_{\sigma_{s'}} + 2\lambda_s \lambda_{s'} [\delta\sigma_s \delta\sigma_{s'}]$$

is a minimum.

Accordingly we have for the minimum value of $\sigma_{\sigma_{ss'}}$ the equation

$$\left.\begin{aligned} \sigma^2_{\sigma_{ss'}} &= \frac{\sigma^2_{\sigma_s}\sigma^2_{\sigma_{s'}} - [\delta\sigma_s\delta\sigma_{s'}]^2}{\sigma^2_{\sigma_s} + \sigma^2_{\sigma_{s'}} - 2[\delta\sigma_s\delta\sigma_{s'}]}, \\ \lambda_s = \frac{\sigma^2_{\sigma_{s'}} - [\delta\sigma_s\delta\sigma_{s'}]}{\sigma^2_{\sigma_s} + \sigma^2_{\sigma_{s'}} - 2[\delta\sigma_s\delta\sigma_{s'}]}, &\quad \lambda_{s'} = \frac{\sigma^2_{\sigma_s} - [\delta\sigma_s\delta\sigma_{s'}]}{\sigma^2_{\sigma_s} + \sigma^2_{\sigma_{s'}} - 2[\delta\sigma_s\delta\sigma_{s'}]} \end{aligned}\right\} \quad \ldots(39).$$

We already know (see p. 119) the values of σ_{σ_s}, $\sigma_{\sigma_{s'}}$ as found from any symmetrical pair of grades, i.e.

$$\sigma^2_{\sigma_s} = \frac{\sigma^2}{2N} \frac{1}{H_s^2 h_s^2} \frac{\bar{n}_s}{N} \left(1 - \frac{2\bar{n}_s}{N}\right).$$

It is easy to show that

$$[\delta\sigma_s \delta\sigma_{s'}] = \sigma_{\sigma_s}\sigma_{\sigma_{s'}} \sqrt{\frac{\frac{2\bar{n}_s}{N}}{1 - \frac{2\bar{n}_s}{N}}} \sqrt{\frac{1 - \frac{2\bar{n}_{s'}}{N}}{\frac{2\bar{n}_{s'}}{N}}} \quad \ldots\ldots\ldots\ldots(40)$$

where $\bar{n}_s$ is always less than $\bar{n}_{s'}$.

It will be seen that the first requisite is to table $r_{\sigma_s \sigma_{s'}}$. We have

TABLE VII. *Correlations of Errors in σ_s and $\sigma_{s'}$, or values of $[\delta\sigma_s\delta\sigma_{s'}]/(\sigma_{\sigma_s}\sigma_{\sigma_{s'}})$.*

$\bar{n}_s/N$ \ $\bar{n}_{s'}/N$	$\frac{1}{40}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{2}$
$\frac{1}{40}$	1·000,000	·906,083	·726,483	·593,171	·484,322	·296,586	·242,161	·197,724	·121,081	·000,000
$\frac{1}{30}$	—	1·000,000	·801,784	654,654	·534,522	·327,327	·267,261	·218,218	·131,815	·000,000
$\frac{1}{20}$	—	—	1·000,000	·816,497	·666,667	·408,248	·333,333	·272,166	·166,667	000,000
$\frac{1}{14}$	—	—	—	1·000,000	·816,497	·500,000	·408,248	·333,333	·204,124	·000,000
$\frac{1}{10}$	—	—	—	—	1·000,000	·612,372	·500,000	·408,248	·250,000	·000,000
$\frac{2}{10}$	—	—	—	—	—	1·000,000	·816,497	·666,667	·408,248	·000,000
$\frac{1}{4}$	—	—	—	—	—	—	1·000,000	·816,496	·500,000	·000,000
$\frac{3}{10}$	—	—	—	—	—	—	—	1·000,000	·612,372	·000,000
$\frac{4}{10}$	—	—	—	—	—	—	—	—	1·000,000	·000,000
$\frac{1}{2}$	—	—	—	—	—	—	—	—	—	1·000,000

TABLE VIII. *Standard Deviations in $\sigma_{\sigma_{ss'}}$, or mean square error in σ as found from two pairs of grades n_s and $n_{s'}$ properly weighted.*

$\bar{n}_s/N$ \ $\bar{n}_{s'}/N$		$\frac{1}{40}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{4}{10}$	
$\frac{1}{40}$	$\sigma_{\sigma_{ss'}}$	1·3453	1·2839	1·2037	1·1471	1·1181	1·1292	1·1582	1·1933	1·2709	$\times\sigma/\sqrt{2N}$
	λ_s	0	·30024	·36974	·41434	·43488	·56284	·63180	·70412	·85474	—
	$\lambda_{s'}$	1·00000	·69976	·63026	·58566	·56512	·43716	·36820	·29588	·14526	—
$\frac{1}{30}$	$\sigma_{\sigma_{ss'}}$	—	1·2954	1·2061	1·1503	1·1175	1·1176	1·1410	1·1701	1·2340	$\times\sigma/\sqrt{2N}$
	λ_s	—	0	·41128	·43490	·46821	·59323	·66010	·72921	·86898	—
	$\lambda_{s'}$	—	1·00000	·58872	·56510	·53179	·40677	·33990	·27079	·13102	—
$\frac{1}{20}$	$\sigma_{\sigma_{ss'}}$	—	—	1·2505	1·1858	1·1448	1·1254	1·1401	1·1606	1·2073	$\times\sigma/\sqrt{2N}$
	λ_s	—	—	0	·47329	·50849	·63438	·69886	·76518	·88961	—
	$\lambda_{s'}$	—	—	1·00000	·62671	·49151	·36562	·30114	·23482	·11039	—
$\frac{1}{14}$	$\sigma_{\sigma_{ss'}}$	—	—	—	1·2383	1·1889	1·1516	1·1594	1·1732	1·2070	$\times\sigma/\sqrt{2N}$
	λ_s	—	—	—	0	·54211	·66755	·72955	·79042	·90467	—
	$\lambda_{s'}$	—	—	—	1·00000	·45789	·33245	·27045	·20958	·09533	—
$\frac{1}{10}$	$\sigma_{\sigma_{ss'}}$	—	—	—	—	1·2576	1·2021	1·2036	1·2117	1·2352	$\times\sigma/\sqrt{2N}$
	λ_s	—	—	—	—	0	·69611	·75568	·82034	·91706	—
	$\lambda_{s'}$	—	—	—	—	1·00000	·30389	·24432	·17966	·08294	—
$\frac{2}{10}$	$\sigma_{\sigma_{ss'}}$	—	—	—	—	—	1·4702	1·4580	1·4552	1·4614	$\times\sigma/\sqrt{2N}$
	λ_s	—	—	—	—	—	0	·80332	·85376	·93936	—
	$\lambda_{s'}$	—	—	—	—	—	1·00000	·19668	·14624	·06064	—
$\frac{1}{4}$	$\sigma_{\sigma_{ss'}}$	—	—	—	—	—	—	1·6495	1·6429	1·6440	$\times\sigma/\sqrt{2N}$
	λ_s	—	—	—	—	—	—	0	·86640	·94646	—
	$\lambda_{s'}$	—	—	—	—	—	—	1·00000	·13360	·05354	—
$\frac{3}{10}$	$\sigma_{\sigma_{ss'}}$	—	—	—	—	—	—	—	1·8999	1·9051	$\times\sigma/\sqrt{2N}$
	λ_s	—	—	—	—	—	—	—	0	·94934	—
	$\lambda_{s'}$	—	—	—	—	—	—	—	1·00000	·05066	—
$\frac{4}{10}$	$\sigma_{\sigma_{ss'}}$	—	—	—	—	—	—	—	—	2·8897	$\times\sigma/\sqrt{2N}$
	λ_s	—	—	—	—	—	—	—	—	0	—
	$\lambda_{s'}$	—	—	—	—	—	—	—	—	1·00000	—

As in the case of the discovery of the best value of the median from two balanced pairs of grades, it will be seen again that there are two factors at work. When the pairs of grades are close to each other then the correlation is high in the errors of σ, and we lose the advantage of duplicating our determinations of σ. On the other hand when the grades are taken far apart the correlation is low and the gain in duplicating our determinations should be greater. But in this case one or other pair of grades must be far removed from the $\frac{1}{14}$ grade which gives the lowest probable error of any determination of σ.

Our second table shows us that we must push one pair of grades considerably out from the $\frac{1}{14}$ pair, and the other pair considerably in. The lowest value occurs at a little under 12 °/₀ increase of inaccuracy above mean and moment values of σ. The pairs of grades are approximately $\frac{1}{30}$ and $\frac{1}{10}$ or $\frac{2}{10}$. As the latter gives very closely the proportions ·6 and ·4 the following rule was tested. Take ·6 times the value of σ as found from the interval between the $\frac{1}{30}$ grades and ·4 times the value of σ as found from the $\frac{1}{8}$ grades (i.e. the second deciles from either end). The mean square error of σ is then found to be $1{\cdot}1175\,\frac{\sigma}{\sqrt{2N}}$ exactly. This is close to the minimum value obtainable by the use of two pairs of grades.

Of course by using three pairs of grades we can still further increase the accuracy of grade methods, but the choice of the three *best* pairs would require lengthy investigations for their determination, and every increase of complexity in grade investigations reduces their already small advantage over mean and moment methods. We may illustrate three pairs on the supposition that $\frac{1}{30}$, $\frac{1}{14}$ and $\frac{2}{10}$ are suitable grade pairs.

We have
$$\sigma_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{10}} = \lambda_{\frac{1}{30}}\sigma_{\frac{1}{30}} + \lambda_{\frac{1}{14}}\sigma_{\frac{1}{14}} + \lambda_{\frac{2}{10}}\sigma_{\frac{2}{10}}$$

leading to

$$\sigma^2_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{10}} = \lambda^2_{\frac{1}{30}}\sigma^2_{\sigma_{\frac{1}{30}}} + \lambda^2_{\frac{1}{14}}\sigma^2_{\sigma_{\frac{1}{14}}} + \lambda^2_{\frac{2}{10}}\sigma^2_{\sigma_{\frac{2}{10}}}$$
$$+ 2\lambda_{\frac{1}{30}}\lambda_{\frac{1}{14}}[\delta\sigma_{\frac{1}{30}}\delta\sigma_{\frac{1}{14}}] + 2\lambda_{\frac{1}{14}}\lambda_{\frac{2}{10}}[\delta\sigma_{\frac{1}{14}}\delta\sigma_{\frac{2}{10}}] + 2\lambda_{\frac{2}{10}}\lambda_{\frac{1}{30}}[\delta\sigma_{\frac{2}{10}}\delta\sigma_{\frac{1}{30}}].$$

Substituting the numerical values from Tables VII and VIII we have the following equation from which to find $\lambda_{\frac{1}{30}}$, $\lambda_{\frac{1}{14}}$, $\lambda_{\frac{2}{10}}$ subject to the condition

$$\lambda_{\frac{1}{30}} + \lambda_{\frac{1}{14}} + \lambda_{\frac{2}{10}} = 1:$$

$$\sigma^2_{\sigma_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{10}}} = \lambda^2_{\frac{1}{30}}\,1{\cdot}6780{,}6116 + \lambda^2_{\frac{1}{14}}\,1{\cdot}5333{,}8689 + \lambda^2_{\frac{2}{10}}\,2{\cdot}1614{,}8804$$
$$+ 2\lambda_{\frac{1}{30}}\lambda_{\frac{1}{14}}\,1{\cdot}0501{,}2591 + 2\lambda_{\frac{1}{14}}\lambda_{\frac{2}{10}}\,{\cdot}9102{,}7433 + 2\lambda_{\frac{2}{10}}\lambda_{\frac{1}{30}}\,{\cdot}6233{,}9299.$$

We determine

$$\lambda_{\frac{1}{30}} = {\cdot}38325, \quad \lambda_{\frac{1}{14}} = {\cdot}32446, \quad \lambda_{\frac{2}{10}} = {\cdot}29229 \quad \text{.....(41),}$$

leading to
$$\sigma_{\sigma_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{10}}} = 1{\cdot}0798\,\frac{\sigma}{\sqrt{2N}} \quad \text{.....(42).}$$

The values of λ are so nearly one-third that without much greater inaccuracy we can assume them $\frac{1}{3}$, we have then

$$\sigma_{\frac{1}{10},\ \frac{1}{14},\ \frac{3}{10}} = \tfrac{1}{3}\left(\sigma_{\frac{1}{10}} + \sigma_{\frac{1}{14}} + \sigma_{\frac{3}{10}}\right) \quad \text{.....(43),}$$

with

$$\sigma_{\sigma_{\frac{1}{10},\ \frac{1}{14},\ \frac{3}{10}}} = 1{\cdot}0822\,\frac{\sigma}{\sqrt{2N}} \quad \text{.....(44).}$$

It is quite conceivable that another choice of the three pairs of grades might lead to an even smaller increase of inaccuracy over the mean and moment method of finding σ. But it suffices here to have shown that the excess inaccuracy by discarding the use of quartiles for quatuordecimals and then these for the $\frac{1}{10}$ and $\frac{3}{10}$ pairs of grades and lastly the latter for three pairs of grades can be reduced from 65 °/₀ to 24 °/₀, from 24 °/₀ to 12 °/₀ and ultimately to 8 °/₀.

(7) In the course of this paper we have seen that it is quite possible to better vastly the determination of median and quartile deviation by not determining the median and quartile directly but from pairs of appropriately chosen grades. We have further seen, however, that either these chosen grades will not be those of classification, or on the other hand if the individual ranking has been obtained as in small series, the measurement of the ranked individuals is hardly likely to be fine enough to give the mean and standard deviation to an adequate number of decimal points. In short while the disadvantages in accuracy are great, the advantages in brevity of treatment will not be compensatory in the majority of cases, and mean and moment methods should be always given the preference. Why then is it needful to discuss at length the probable errors of grade methods? The answer lies in the fact that in all cases of classification by broad qualitative categories no other method of solution is feasible. Further it is certain that hitherto too little attention has been paid to the best means of deducing the constants of such distributions. In such cases it is futile to attempt to obtain directly median quartiles or deciles. Such attempts throw us back on interpolations of an unsatisfactory character. We are compelled to accept the grades given by the data themselves, and these grades will usually not be symmetrical and are not capable of satisfactory readjustment. From every pair of grades we can find the standard deviation in terms of a chosen variate interval, and from every grade we can find the position of the mean in terms of this standard deviation. The standard deviations as found from each pair of grades will not, however, be independent; if there be p broad categories only $p-2$ values of the standard deviation in terms of the variate sub-ranges are *a priori* open to choice and only $p-1$ determinations of the median. Let us consider how this, the really practical problem of grading, is associated with the methods discussed in this paper.

We start with the sampled population, the mean of which may be treated as a fixed origin for measurements in the samples. This population is divided up by dichotomic ordinates into broad categories; the distances between these dichotomic ordinates will be the same in the sample as for the sampled population, and although their values have to be found from the sample they are really constant

differences depending on the classification chosen. The essential feature of the sample is that the frequency found between these dichotomic ordinates is not exactly the proportion to be expected on the average, i.e. between x_s and x_{s+1} it is not $\bar{n}_{s+1} - \bar{n}_s$ but differs from this by $\delta n_{s+1} - \delta n_s$, where δn_s, δn_{s+1} are the deviations of random sampling*.

Let $\bar{h}_r$, $\bar{x}_r$, $\bar{\sigma}$ be quantities of the sampled population and h_r, x_r, σ refer to the sample. Further let σ_{rs} be the σ of the sample as found from the rth and sth dichotomic interval. Thus if $x_r > x_s$ or $r < s$, measuring x positive to the right-hand, we have

$$(h_r - h_s)\,\sigma_{rs} = x_r - x_s = \bar{x}_r - \bar{x}_s = \sigma\,(\bar{h}_r - \bar{h}_s).$$

Accordingly to a first approximation:

$$\frac{\delta\sigma_{rs}}{\bar{\sigma}} = -\frac{\delta h_r - \delta h_s}{\bar{h}_r - \bar{h}_s},$$

or, $$\sigma^2_{\sigma_{rs}} = \frac{\bar{\sigma}^2}{(\bar{h}_r - \bar{h}_s)^2}\{\sigma^2_{h_r} + \sigma^2_{h_s} - 2\,[\delta h_r \delta h_s]\}.$$

Thus

$$\sigma_{\sigma_{rs}} = \frac{\bar{\sigma}}{\sqrt{2N}}\frac{1}{\bar{h}_r - \bar{h}_s}\left\{\frac{2}{\bar{H}_r^2}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_r}{N}\right) + \frac{2}{\bar{H}_s^2}\frac{\bar{n}_s}{N}\left(1 - \frac{\bar{n}_s}{N}\right) - \frac{4}{\bar{H}_r\bar{H}_s}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_s}{N}\right)\right\}^{\frac{1}{2}} \quad \ldots\ldots(45).$$

Now it is convenient to put this in a somewhat different form. Let two dichotomic ordinates be taken and suppose the range between them to represent 100 units of the character in question. It is convenient to call this unit in general a *notace*, while *mentace, sanitace, munditace* represent special units of intelligence, health, cleanliness, etc. Let these dichotomic ordinates be p and q, $p < q$.

Then $100 = \bar{\sigma}\,(\bar{h}_p - \bar{h}_q)$ and accordingly:

$$\sigma_{\sigma_{rs}} = \frac{100}{\sqrt{2N}}\frac{1}{(\bar{h}_p - \bar{h}_q)(\bar{h}_r - \bar{h}_s)}\left\{\frac{2}{\bar{H}_r^2}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_r}{N}\right) + \frac{2}{\bar{H}_s^2}\frac{\bar{n}_s}{N}\left(1 - \frac{\bar{n}_s}{N}\right) - \frac{4}{\bar{H}_r\bar{H}_s}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_s}{N}\right)\right\}^{\frac{1}{2}} \quad \ldots\ldots(46).$$

In actually calculating this value of the $\sigma_{\sigma_{rs}}$ as found from the rth and sth dichotomic ordinates we must of course substitute for $\bar{h}_p$, $\bar{h}_q$, $\bar{h}_r$, $\bar{h}_s$, $\bar{H}_p$, $\bar{H}_q$, $\bar{n}_r$, $\bar{n}_s$ the values provided by the sample in the usual way.

Now it will be clear that it is not easy to determine which pair of dichotomic ordinates will give the best, i.e. lowest, value of $\sigma_{\sigma_{rs}}$. For while it is disadvantageous to select s near to r owing to the presence of $\bar{h}_r - \bar{h}_s$ in the denominator, the increasing correlation of h_r and h_s as s approaches r makes the radical smaller and smaller.

Our previous investigations, however, indicate to us the neighbourhood where $\sigma_{\sigma_{rs}}$ is a minimum and we shall endeavour to choose our pair of grades out of the

* We suppose the subscript s to refer to the sth dichotomic ordinate numbered from the right-hand end of the distribution.

predetermined dichotomic ordinates to approach these as far as the material permits. As a rule $\bar{h}_s$ (and $\bar{h}_q$) will have the opposite sign to $\bar{h}_r$ (and $\bar{h}_p$), but it will not be possible to make them of equal magnitude.

It will probably be desirable to work with several pairs of dichotomic lines. In this case $\sigma_{\sigma_{r's'}}$ can be written down at once from the value of $\sigma_{\sigma_{rs}}$ by proper interchange of subscripts. But we need $[\delta\sigma_{rs}\delta\sigma_{r's'}]$. If we suppose the r' and s' dichotomic ordinates to fall on the median sides of r and s respectively, we easily find in the manner of earlier work in this paper:

$$[\delta\sigma_{rs}\delta\sigma_{r's'}] = \frac{100^2}{(\bar{h}_p-\bar{h}_q)^2(\bar{h}_r-\bar{h}_s)(\bar{h}_{r'}-\bar{h}_{s'})}\left\{\frac{1}{\bar{H}_r\bar{H}_{r'}}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_{r'}}{N}\right)-\frac{1}{\bar{H}_r\bar{H}_{s'}}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_{s'}}{N}\right)\right.$$
$$\left.-\frac{1}{\bar{H}_s\bar{H}_{r'}}\frac{\bar{n}_{r'}}{N}\left(1-\frac{\bar{n}_s}{N}\right)+\frac{1}{H_s\bar{H}_{s'}}\frac{\bar{n}_{s'}}{N}\left(1-\frac{\bar{r}_s}{N}\right)\right\} \quad \text{.....(47).}$$

Further any range like $x_r - x_s$ is to be found from

$$x_r - x_s = 100\,(\bar{h}_r-\bar{h}_s)/(\bar{h}_p-\bar{h}_q) \text{.....(48),}$$

and the error in $x_r - x_s$ will arise from our using h_r, h_s, h_p, h_q instead of $\bar{h}_r$, $\bar{h}_s$, $\bar{h}_p$, $\bar{h}_q$. In other words:

$$\frac{\delta(x_r - x_s)}{x_r - x_s} = \frac{\delta h_r - \delta h_s}{\bar{h}_r - \bar{h}_s} - \frac{\delta h_p - \delta h_q}{\bar{h}_p - \bar{h}_q}$$
$$= \frac{\delta\sigma_{pq}}{\bar{\sigma}} - \frac{\delta\sigma_{rs}}{\bar{\sigma}},$$

or
$$\sigma^2_{(x_r-x_s)} = (\bar{h}_r-\bar{h}_s)^2\{\sigma^2_{\sigma_{pq}} + \sigma^2_{\sigma_{rs}} - 2\,[\delta\sigma_{pq}\delta\sigma_{rs}]\}.$$

The terms in the curled brackets are all known from equations (46) and (47) above, and we can thus determine the probable error of each portion $x_r - x_s$ of our notace scale as found from equation (48).

We can now consider the position of the mean. The distance of the mean from the rth dichotomic line is

$$\bar{x}_r = \frac{\bar{h}_r 100}{\bar{h}_p - \bar{h}_q},$$

and accordingly
$$\frac{\delta x_r}{\bar{x}_r} = \frac{\delta h_r}{\bar{h}_r} - \frac{\delta h_p - \delta h_q}{\bar{h}_p - \bar{h}_q},$$

or
$$\delta x_r = \bar{\sigma}\delta h_r + \bar{x}_r\delta\sigma_{pq}/\bar{\sigma}.$$

Thus
$$\sigma^2_{x_r} = \bar{\sigma}^2\sigma^2_{h_r} + \bar{x}_r{}^2\sigma^2_{\sigma_{pq}} + 2\bar{x}_r\bar{\sigma}\,[\delta h_r\delta\sigma_{pq}] \text{.....(49).}$$

Now this may be written:

$$\sigma^2_{x_r} = \bar{\sigma}^2\left\{\sigma^2_{h_r} + \bar{x}_r{}^2\frac{\sigma^2_{\sigma_{pq}}}{\sigma^2} + 2\bar{h}_r\left[\delta h_r\frac{\delta\sigma_{pq}}{\sigma}\right]\right\} \text{.....(49 \textit{bis}).}$$

Now
$$\sigma^2_{h_r} = \frac{1}{NH_r{}^2}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_r}{N}\right) \text{.....(50).}$$

$\sigma^2_{\sigma_{pq}}$ is given in equation (46) and we easily find

$$\left[\delta h_r \frac{\delta \sigma_{pq}}{\sigma}\right] = -\frac{1}{N}\frac{1}{\bar{h}_p - \bar{h}_q}\left\{\frac{1}{\bar{H}_p \bar{H}_r}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_p}{N}\right) - \frac{1}{\bar{H}_q \bar{H}_r}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_q}{N}\right)\right\}$$

$$\text{if } \bar{n}_r < \bar{n}_p \text{ and } < \bar{n}_q,$$

$$= -\frac{1}{N}\frac{1}{\bar{h}_p - \bar{h}_q}\left\{\frac{1}{\bar{H}_p \bar{H}_r}\frac{\bar{n}_p}{N}\left(1-\frac{\bar{n}_r}{N}\right) - \frac{1}{\bar{H}_q \bar{H}_r}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_q}{N}\right)\right\}$$

$$\text{if } \bar{n}_r > \bar{n}_p \text{ and } < \bar{n}_q,$$

$$= -\frac{1}{N}\frac{1}{\bar{h}_p - \bar{h}_q}\left\{\frac{1}{\bar{H}_p \bar{H}_r}\frac{\bar{n}_p}{N}\left(1-\frac{\bar{n}_r}{N}\right) - \frac{1}{\bar{H}_q \bar{H}_r}\frac{\bar{n}_q}{N}\left(1-\frac{\bar{n}_r}{N}\right)\right\}$$

$$\text{if } \bar{n}_r > \bar{n}_p \text{ and } > \bar{n}_q \quad \ldots\ldots\ldots\ldots (51).$$

$\sigma^2_{x_r}$ can always be found, and the error of our determination of the mean from the rth dichotomic line determined. But unless we are given some lengths on our variate line corresponding to dichotomic lines we can only choose (i) the p, q range to give σ, (ii) the rth dichotomic line to determine the mean. All other scale lines are determined by these, but the degree of accuracy of their determination depends on the fitting choice of p, q and r.

(8) As illustration let us determine the probable errors of the following distribution of Intelligence in Schoolboys on the assumption that the distribution may be taken as normal.

Quick Intelligent	279·5	(x_5, h_5)	$n_5/N = 1{\cdot}000{,}000$
Intelligent	788	(x_4, h_4)	$n_4/N = \cdot 883{,}005$
Slow Intelligent	771·5	(x_3, h_3)	$n_3/N = \cdot 553{,}160$
Slow	369	(x_2, h_2)	$n_2/N = \cdot 230{,}222$
Slow Dull	139·5	(x_1, h_1)	$n_1/N = \cdot 075{,}764$
Very Dull	41·5		$n_1/N = \cdot 017{,}371$
	2389		

$H_1 = \cdot 042{,}944$, $H_2 = \cdot 142{,}542$, $H_3 = \cdot 303{,}847$, $H_4 = \cdot 395{,}391$, $H_5 = \cdot 196{,}487$

$h_1 = 2{\cdot}111{,}366$, $h_2 = 1{\cdot}434{,}202$, $h_3 = \cdot 737{,}957$, $h_4 = -\cdot 133{,}651$, $h_5 = -1{\cdot}190{,}144$

Taking the range of Intelligent to be 100 mentaces we have:

$$\sigma(h_5 - h_4) = 100, \text{ or } \sigma = 94{\cdot}653 \text{ mentaces.}$$

Whence by (46), p. 129, putting $r = p$, $s = q$, we have

$$\sigma_{\sigma_{5,4}} = 2{\cdot}4594 \text{ mentaces,}$$

and the probable error of $\sigma_{5,4} = 1{\cdot}6589$ mentaces or is less than two mentaces, which is amply accurate for this type of work.

The position of the mean is at $h_4\sigma$ from the border of Intelligent and Slow Intelligent into the latter = 12·6504 mentaces from the 4th dichotomic line.

We want the probable error of such a value. It equals $\frac{h_4 100}{h_5 - h_4} = x_4$, whence

$$\frac{\sigma^2_{x_4}}{\bar{x}_4^2} = \frac{\sigma^2_{h_4}}{\bar{h}_4^2} + \frac{1}{(\bar{h}_5 - \bar{h}_4)^2}(\sigma^2_{h_5} + \sigma^2_{h_4} - 2[\delta h_5 \delta h_4]) - \frac{2}{\bar{h}_4(\bar{h}_5 - \bar{h}_4)}([\delta h_4 \delta h_5] - \sigma^2_{h_4})$$

$$= \frac{1}{\bar{h}_4^2(\bar{h}_5 - \bar{h}_4)^2}(\sigma^2_{h_4}\bar{h}_5^2 + \sigma^2_{h_5}\bar{h}_4^2 - 2\bar{h}_4\bar{h}_5[\delta h_4 \delta h_5]).$$

Thus

$$\sigma_{x_4} = \frac{100}{(\bar{h}_5 - \bar{h}_4)^2}(\sigma^2_{h_4}\bar{h}_5^2 + \sigma^2_{h_5}\bar{h}_4^2 - 2\bar{h}_4\bar{h}_5[\delta h_4 \delta h_5])^{\frac{1}{2}} \quad \text{.....(52),}$$

$$= \frac{1}{\sqrt{N}} \frac{100}{(\bar{h}_5 - \bar{h}_4)^2}\left\{\frac{\bar{h}_5^2}{\bar{H}_4^2}\frac{\bar{n}_4}{N}\left(1 - \frac{\bar{n}_4}{N}\right) + \frac{\bar{h}_4^2}{\bar{H}_5^2}\frac{\bar{n}_5}{N}\left(1 - \frac{\bar{n}_5}{N}\right) - \frac{2\bar{h}_4\bar{h}_5}{\bar{H}_4\bar{H}_5}\frac{\bar{n}_4}{N}\left(1 - \frac{\bar{n}_5}{N}\right)\right\}^{\frac{1}{2}}$$

.....(52 *bis*).

Applying this to our case we have

$$\sigma_{x_4} = 2{\cdot}759{,}657,$$

or the probable error of the position of the mean from the 4th dichotomic line is 1·8614 mentaces. Symmetry of the result, as we might also anticipate from *a priori* reasons, shows us that this is also the probable error of the distance of the mean from the 5th dichotomic line.

We may conclude this investigation by considering the error in the position of the 2nd dichotomic line, or that which divides the Slow and Slow Dull categories.

We have $\bar{x}_2 = \frac{\bar{h}_2 100}{\bar{h}_5 - \bar{h}_4} = 135{\cdot}7512$ mentaces,

and using (49) and (51) obtain

$$\sigma_{x_2} = 4{\cdot}4241,$$

or the probable error of x_2 is 2·9840 mentaces.

Thus in terms of our Intelligent Range as 100 mentaces we find in mentaces

$$\begin{aligned} x_2 &= 135{\cdot}7512 \pm 2{\cdot}9840, \\ x_4 &= -\ 12{\cdot}6504 \pm 1{\cdot}8614, \\ x_5 &= -112{\cdot}6504 \pm 1{\cdot}8614, \\ \sigma_{4,5} &= 94{\cdot}653 \pm 1{\cdot}659. \end{aligned}$$

Any other dichotomic ordinate can have its probable error determined in like manner.

It is clear that the positions of the dichotomic lines can be found with probable errors of 2 to 3 mentaces. The whole range of intelligence being about 600 mentaces we see that with 2389 individuals, we can determine the positions of the category divisions within ranges of about 5 to 8 mentaces in a total range of variation of 600 mentaces.